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Differential Algebraic Equations
Exercise Sheet 4 – One Step Methods

We consider the initial value problems

$$\dot{x} = f(t, x), \quad x(t_0) = x_0, \tag{ODE}$$

and

$$F(t, x, \dot{x}) =, \quad x(t_0) = x_0, \tag{DAE}$$

on the time interval $[t_0, T]$. On the grid

$$t_0 < t_1 < \cdots < t_N,$$

where $N \in \mathbb{N}$, where $t_k = t_0 + kh$, and where $h = \frac{T - t_0}{N}$, let x_k denote a numerical approximation to $x(t_k)$, where x is a solution to (ODE) or (DAE). We use the general form

$$\mathfrak{X}_{k+1} = \mathfrak{F}(t_k, \mathfrak{X}_k, h)$$

to refer to a given numerical scheme.

A Effect of Rounding Errors Consider the *Explicit Euler* scheme

$$x_{k+1} = x_k + hf(t_k, x_k)$$

and show that the error $e_k = x_k - y_k$, where y_k are the iterates of the perturbed scheme

$$y_{k+1} = y_k + hf(t_k, y_k) + \varepsilon_k,$$

and where ε_k denotes the rounding error, behaves like $\|e_N\| \approx \max_k\{\epsilon_k\}h^{-1}$.

B Order of Consistency Determine the order of consistency of the Runge–Kutta scheme

$$\begin{array}{c|cc} 0 & 0 & \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & 0 & 1 \end{array}$$

by directly estimating $\|\mathfrak{X}(t_{k+1}) - \mathfrak{F}(t_{k+1}, \mathfrak{X}(t_{k+1}), h)\|$.

C Two-stage Gauss method for ODEs and DAEs Consider the *Gauss* method of order 2:

$$\begin{array}{c|cc} \frac{1}{2} - \frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} \\ \frac{1}{2} + \frac{\sqrt{3}}{6} & \frac{1}{4} + \frac{\sqrt{3}}{6} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

1. Determine the order of convergence for ODEs using *Butcher's Theorem* (lecture: Thm. 5.9).
2. Show that $\kappa_1 = 2$ and $\kappa_2 = 2$ as defined in lecture: Thm. 5.10. What does this mean for the numerical approximation of DAEs?

D Kronecker and Runge–Kutta Show that for the DAE with constant coefficients $E\dot{x} = Ax + f(t)$, the stage derivatives $\dot{X}_{ij}$, $j = 1, \ldots, s$ at time step i of an s-stage Runge–Kutta method $(\mathcal{A}, \beta, \gamma)$, are defined through the linear system

$$(I_s \otimes E - h\mathcal{A} \otimes A)\dot{X}_i = Z_i,$$

where $\dot{X}_i := [\dot{X}_{il}]_{l=1,\ldots,s}$ and $Z_i = [Ax_i + f(t_i + \gamma_l h)]_{l=1,\ldots,s}$.

Please turn the sheet.

Coding Exercises

1. Implement the *explicit Euler* scheme for constant step sizes and test it on the ODE:

$$\begin{aligned} \dot{x}_1 &= e^t x_2, & x_1(0) &= \sin(1), \\ \dot{x}_2 &= -e^t x_1, & x_2(0) &= \cos(1), \end{aligned}$$

   through numerically computing $u_1(3)$ for stepsizes $h = 3/2^k$, $k = 5, 6, \dots$. What do you observe?

2. Implement the *implicit Euler* scheme for DAEs for constant step sizes as a method that takes
   - a function $F(t, x, \dot{x})$,
   - an initial value $x(t_0) = x_0$,
   - an interval $\mathbb{I} = [t_0, T]$,
   - and a number of discretization points N.

   It should return an array of the computed solution and a plot of it.

   Test your implementation on the following cases:

   (a) $F = E\dot{x} - Ax - f$, $x_0 = [0, 1]^T$, $\mathbb{I} = [0, 10]$, and

$$E(t) = \begin{bmatrix} -t & t^2 \\ -1 & t \end{bmatrix}, \quad A(t) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad f(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

   (b) $F = E\dot{x} - Ax - f$, $x_0 = [1, 1]^T$, $\mathbb{I} = [0, 10]$, and

$$E(t) = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \quad A(t) = \begin{bmatrix} -1 & t \\ 0 & 0 \end{bmatrix}, \quad f(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}.$$

   (c) $F = E\dot{x} - Ax - f$, $x_0 = [0, 1]^T$, $\mathbb{I} = [0, 10]$, and

$$E(t) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A(t) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad f(t) = \begin{bmatrix} e^t + \cos(t) \\ e^t \end{bmatrix}.$$

   Compare the outcomes to the actual solutions (cf. the introducing examples of Section 4 in the lecture). Are the initial values consistent? If not, determine consistent ones? What happens for inconsistent initial values?