Otto von Guericke Universität Magdeburg
Fakultät für Mathematik
Jan Heiland

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Differential Algebraic Equations

Exercise Sheet 5 – Higher index DAEs and higher order RKM

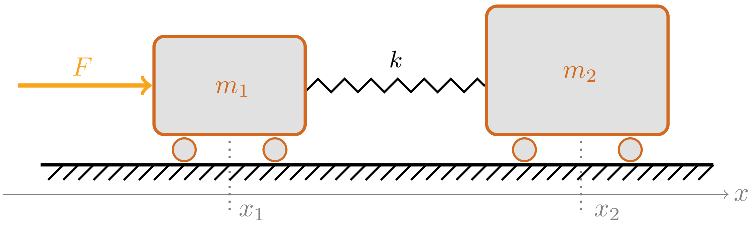

Figure 1: Illustration of a 2-body mass-spring chain moved by an input force F.

We consider the DAE in the variables x_1, x_2, and F,

$$m_1\ddot{x}_1(t) = \quad k\big(x_2(t) - x_1(t) - 0.5\big) + F(t), \tag{1a}$$

$$m_2\ddot{x}_2(t) = -k\big(x_2(t) - x_1(t) - 0.5\big), \tag{1b}$$

$$x_2(t) = \quad g(t), \tag{1c}$$

which derives from the task to steer a mass m_2 along a trajectory g via acting on a connected mass m_1 through an unknown force F; cf. Figure 1. The connection is given by a spring with a constant k.

A Solution and the Index of the State Equations

Transfer the system (1) to a first order system $E\dot{z} = Az + f$ and determine the index of the resulting matrix pair (E, A). What does this mean for the target trajectory g? Derive F analytically.

B Index Reduction by Minimal Extension

Use the approach of *Minimal Extension* to derive an equivalent representation of (1) with a lower index.

C *Radau IIa* Schemes The 1- and the 2-stages *Radau IIa* schemes are given as

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array} \qquad \text{and} \qquad \begin{array}{c|cc} \frac{1}{3} & \frac{5}{12} & -\frac{1}{12} \\ 1 & \frac{3}{4} & \frac{1}{4} \\ \hline & \frac{3}{4} & \frac{1}{4} \end{array} \quad .$$

Show that for the 2-stage scheme, the constants defined in Theorem 5.10 are given as $\kappa_1 = \infty$ and $\kappa_2 = 2$.

Please turn the sheet.

Coding Exercises Implement a numerical time stepping scheme as described in Exercise 4. (You may use and extend the implementation that is for download on the website).

1. Use the *Implicit Euler* scheme to integrate the equations modelling the evolution of the pendulum:

$$m\ddot{x} = -2(x - r_x)\lambda, \tag{2a}$$
$$m\ddot{y} = -2(y - r_y)\lambda - mg, \tag{2b}$$
$$0 = (x - r_x)^2 + (y - r_y)^2 - l^2, \tag{2c}$$

   for suitable parameters m, $r = (r_x, r_y)$, l, and g and suitable initial positions and velocities.

2. Use the *Implicit Euler* scheme to integrate (2) in the (theoretically) equivalent reformulation, where (2c) is replaced by

$$0 = 2(x - r_x)\dot{x} + 2(y - r_y)\dot{y}.$$

   Evaluate the actual constraint (2c) at the computed values and interprete your observations.

3. Use the 2-stage *Radau IIa* scheme to integrate the equations modelling the mass-spring chain (1), with the following parameters

$$m_1 = 2kg, \qquad m_2 = 1kg, \qquad k = 1\tfrac{N}{m}, \qquad d = 0.5m.$$

   initial conditions

$$x_1(0) = 0m,\ \dot{x}_1(0) = 0\tfrac{m}{s}, \qquad x_2(0) = 0.5m,\ \dot{x}_2(0) = 0\tfrac{m}{s}.$$

   and the target trajectory defined via the start and $g_0 = 0.5m$ terminal positions $g_f = 2.5m$, and the manoeuvre $[0, 4s]$ via

$$g(t) = \begin{cases} g_0, & \text{if } 0 \le t < 1, \\ g_0 + p\left(\frac{t-1}{2}\right)(g_f - g_0), & \text{if } 1 \le t \le 3, \\ g_f, & \text{if } 3 < t \le 4, \end{cases}$$

   with the polynomial

$$p(s) = 1716s^7 - 9009s^8 + 20020s^9 - 24024s^{10} + 16380s^{11} - 6006s^{12} + 924s^{13}.$$

4. Use the 2-stage *Radau IIa* and the scheme to integrate the equations modelling the mass-spring chain (1) having applied the index reduction of **B**.

5. Use the 2-stage *Radau IIa* scheme to integrate the equations (2) modelling the evolution of the pendulum.