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Differential Algebraic Equations

Exercise Sheet 5 – Higher index DAEs and higher order RKM

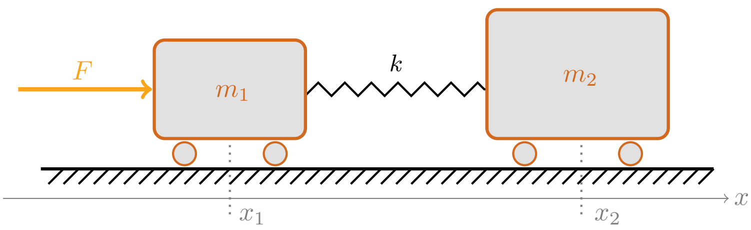

Figure 1: Illustration of a 2-body mass-spring chain moved by an input force F.

We consider the DAE in the variables x_1, x_2, and F,

$$m_1\ddot{x}_1(t) = k\big(x_2(t) - x_1(t) - 0.5\big) + F(t), \tag{1a}$$

$$m_2\ddot{x}_2(t) = -k\big(x_2(t) - x_1(t) - 0.5\big), \tag{1b}$$

$$x_2(t) = g(t), \tag{1c}$$

which derives from the task to steer a mass m_2 along a trajectory g via acting on a connected mass m_1 through an unknown force F; cf. Figure 1. The connection is given by a spring with a constant k.

A Solution and the Index of the State Equations
Transfer the system (1) to a first order system $E\dot{z} = Az + f$ and determine the index of the resulting matrix pair (E, A). What does this mean for the target trajectory g? Derive F analytically.

B Index Reduction by Minimal Extension
Use the approach of *Minimal Extension* to derive an equivalent representation of (1) with a lower index.

C *Radau IIa* Schemes The 1- and the 2-stages *Radau IIa* schemes are given as

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array} \qquad \text{and} \qquad \begin{array}{c|cc} \frac{1}{3} & \frac{5}{12} & -\frac{1}{12} \\ 1 & \frac{3}{4} & \frac{1}{4} \\ \hline & \frac{3}{4} & \frac{1}{4} \end{array}\ .$$

Show that for the 2-stage scheme, the constants defined in Theorem 5.10 are given as $\kappa_1 = \infty$ and $\kappa_2 = 2$.

D *BDF* for higher index problems In the lecture we learned that, in theory, BDF schemes applied to linear DAEs with constant coefficients are convergent of the same order as for ODEs. In practice, the rounding error may affect the convergence. This is generally true for ODEs (see Exercise IV.A) but worse the higher the index of the DAE is. Investigate the propagation of the rounding error in the BDF scheme applied to

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \dot{x} = x + f.$$

Please turn the sheet.

Coding Exercises Implement a numerical time stepping scheme as described in Exercise 4. (You may use and extend the implementation that is for download on the website).

1. Use the *Implicit Euler* scheme to integrate the equations modelling the evolution of the pendulum:

$$
\begin{aligned}
m\ddot{x} &= -2(x - r_x)\lambda, && \text{(2a)}\\
m\ddot{y} &= -2(y - r_y)\lambda - mg, && \text{(2b)}\\
0 &= (x - r_x)^2 + (y - r_y)^2 - l^2, && \text{(2c)}
\end{aligned}
$$

   for suitable parameters m, $r = (r_x, r_y)$, l, and g and suitable initial positions and velocities.

2. Use the *Implicit Euler* scheme to integrate (2) in the (theoretically) equivalent reformulation, where (2c) is replaced by

$$0 = 2(x - r_x)\dot{x} + 2(y - r_y)\dot{y}.$$

   Evaluate the actual constraint (2c) at the computed values and interprete your observations.

3. Use the 2-stage *Radau IIa* scheme to integrate the equations modelling the mass-spring chain (1), with the following parameters

$$m_1 = 2kg, \qquad m_2 = 1kg, \qquad k = 1\tfrac{N}{m}, \qquad d = 0.5m.$$

   initial conditions

$$x_1(0) = 0m,\ \dot{x}_1(0) = 0\tfrac{m}{s}, \qquad x_2(0) = 0.5m,\ \dot{x}_2(0) = 0\tfrac{m}{s}.$$

   and the target trajectory defined via the start and $g_0 = 0.5m$ terminal positions $g_f = 2.5m$, and the manoeuvre $[0, 4s]$ via

$$g(t) = \begin{cases} g_0, & \text{if } 0 \le t < 1,\\ g_0 + p\big(\tfrac{t-1}{2}\big)(g_f - g_0), & \text{if } 1 \le t \le 3,\\ g_f, & \text{if } 3 < t \le 4, \end{cases}$$

   with the polynomial

$$p(s) = 1716s^7 - 9009s^8 + 20020s^9 - 24024s^{10} + 16380s^{11} - 6006s^{12} + 924s^{13}.$$

4. Use the 2-stage *Radau IIa* and the scheme to integrate the equations modelling the mass-spring chain (1) having applied the index reduction of **B**.

5. Use the 2-stage *Radau IIa* scheme to integrate the equations (2) modelling the evolution of the pendulum.