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# Finite Element Decompositions for Stable Time Integration of Flow Equations

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# Problem Statement

- We want to integrate the semi-discrete flow equations, to solve for the velocity $v$ and the pressure $p$:

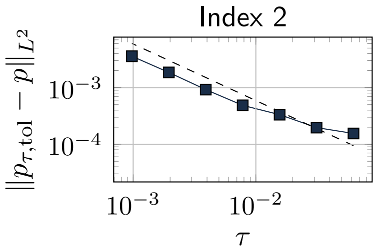
$$\begin{aligned} M\dot{v} + K(v) + B^T p &= f \\ Bv \qquad\qquad &= g \end{aligned}$$

- Why not simply use semi-implicit Euler scheme, which is of first order for $v$ and $p$?

$$\begin{aligned} M\tfrac{v^+ - v^c}{\tau} + K(v^c) + B^T p^c &= f^c \\ Bv^+ \qquad\qquad &= g^+ \end{aligned}$$

# Numerical Example

The error in the pressure variable $p$ for varying $\tau$




if we iteratively solve the resulting linear equation systems up to a residual of $\mathrm{tol} = 10^{-3}$.

# Introduction

- Flow equations

$$\begin{aligned} \dot{u} + \mathcal{K}u + \nabla \pi &= f, \\ \nabla \cdot u &= g, \\ u(0) &= u_0. \end{aligned}$$

- Stokes equation: $\mathcal{K}u = -\nu \Delta u$
- Euler equation: $\mathcal{K}u = (u \cdot \nabla)u$
- Navier-Stokes equations: $\mathcal{K}u = (u \cdot \nabla)u - \nu \Delta u$

# Minimal Extension for the PDE

$$\dot u + \mathcal{K}u + \nabla\pi = f$$
$$\nabla\cdot u = g$$

- We can decompose the solution $u = u_1 + u_2$ such that

$$\nabla\cdot u_1 = 0 \quad \text{and} \quad \nabla\cdot u_2 = g.$$

- We add the derivative of the constrained $\nabla\cdot u = g$ to the system

$$\begin{aligned} \dot u + \mathcal{K}u + \nabla\pi &= f, \\ \nabla\cdot u &= g, \\ \nabla\cdot \dot u &= \dot g. \end{aligned}$$

# Minimal Extension for the PDE

$$\begin{aligned} \dot u + \mathcal{K}u + \nabla \pi &= f \\ \nabla \cdot u &= g \\ \nabla \cdot \dot u &= \dot g \end{aligned}$$

With $u = u_1 + u_2$, $\nabla \cdot u_1 = 0$, and $\tilde u_2 := \dot u_2$, one obtains

$$\begin{aligned} \dot u_1 + \tilde u_2 + \mathcal{K}\big(u_1 + u_2\big) - \nabla \pi &= f \\ \nabla u_2 \qquad &= g \\ \nabla \tilde u_2 \qquad\qquad\qquad &= \dot g \end{aligned}$$

which

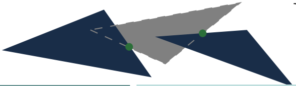
- – under reasonable conditions – is *equivalent* to the original system [Altmann '15, Altmann/JH '15]
- is an *index reduced* on the PDE level applying Minimal Extension [Kunkel/Mehrmann '04]

# Spatial Discretization

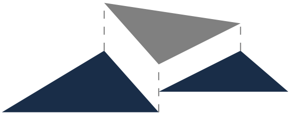
$$\dot{u} + \mathcal{K}u + \nabla\pi = f$$
$$\nabla \cdot u = g$$

- Velocity: $u \in \mathcal{V} := \left[H_0^1(\Omega)\right]^d \approx v \in V_h$
- Pressure: $\pi \in \mathcal{Q} := L^2(\Omega)/\mathbb{R} \approx p \in Q_h$

- $V_h$ and $Q_h$ satisfy an *inf-sup* or the *LBB* condition
- like the *Taylor-Hood* or *Crouzeix-Raviart* scheme



$V_h$



$Q_h$

# Spatial Discretization

$$\dot{u} + \mathcal{K}u + \nabla\pi = f$$
$$\nabla \cdot u = g$$

$$M\dot{v} + K(v) + B^T p = f$$
$$Bv = g \quad (*)$$

- Assumptions on the DAE (met by *LBB-stable* schemes)
  - $M \in \mathbb{R}^{n_v,n_v}$, $B \in \mathbb{R}^{n_p,n_v}$, $n_v > n_p$
  - $M$ symmetric positive definite and $B$ is of full rank
- Proposed reformulation
  - Add the time-derivative of the constraint $(*)$
  - Add a variable that makes the system square again

# Reformulation

$$\begin{aligned} M\dot v + K(v) + B^T p &= f \\ Bv \qquad\qquad &= g \quad (*) \end{aligned}$$

- Find orthogonal matrix $Q$ so that
  - $BQ = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$ with invertible $B_2$
  - and define $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} := Q^T v$
- Add constraint $B\dot q = \dot g$ and the new variable $\tilde q_2 := \dot q_2$

# Reformulation

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Reformulated "index-1" system:

$$\begin{aligned} MQ \begin{bmatrix} \dot q_1 \\ \tilde q_2 \end{bmatrix} + K(q_1, q_2) + B^T p &= f \\ B_1 q_1 + B_2 q_2 &= g \\ B_1 \dot q_1 + B_2 \tilde q_2 &= \dot g \end{aligned}$$

# Finite Element Decomposition

- Problem: how to find $Q$
  - Computation via QR decomposition is not feasible
- We propose: decomposition of velocity ansatz space
  - Splitting $V_h = V_{h,1} \oplus V_{h,2}$
  - Basis functions $V_{h,1} = \text{span}\{\varphi_1, \dots, \varphi_{n_v-m}\}$,
    $V_{h,2} = \text{span}\{\varphi_{n_v-m+1}, \dots, \varphi_{n_v}\}$
  - $Q =$ permutation matrix
  - $B \to \begin{bmatrix} B_1 & B_2 \end{bmatrix}$ through column swapping
- **Aim**: find a subspace $V_{h,2}$, or $\varphi_{n_v-m+1}, \dots, \varphi_{n_v}$, such that $B_2$ is invertible

# Finite Element Decomposition

- Algorithm to find $V_{h,2}$ are available for
  - Crouzeix-Raviart elements,
  - Taylor-Hood elements,
  - Rannacher-Turek elements for quads,
  - and their 3D counterparts.

  see [Altmann/JH '15].

- Illustration for the Crouzeix-Raviart scheme
  - The splitting is based on a mapping $\iota \colon \mathcal{T} \setminus \{T_0\} \to \mathcal{E}$
  - Range$(\iota) = \{E_1, \dots, E_m\}$ subset of the set of edges $\mathcal{E}$ of the triangulation
  - Edges correspond to $\varphi_{n-m+1}, \dots, \varphi_n$
  - ...
  - The algorithm terminates in a triangular $B_2$

# Time Integration

- Consider the semi-explicit Euler scheme
  - $\tau$ . . . time-step length
  - $v^c$, $v^+$ . . . current, next time instance of velocity

- For the index-2 system:

$$\begin{aligned} M\tfrac{v^+ - v^c}{\tau} + K(v^c) + B^T p^c &= f^c \\ Bv^+ \qquad\qquad &= g^+ \end{aligned}$$

- For the index-1 system with $(q_1, q_2) \leftarrow Q^T v$:

$$\begin{aligned} MQ\begin{bmatrix} \frac{1}{\tau}(q_1^+ - q_1^c) \\ \tilde{q}_2^c \end{bmatrix} + K(q_1^c, q_2^c) + B^T p^c &= f^c \\ B_1 q_1^+ + B_2 q_2^+ \qquad &= g^+ \\ B_1 \tfrac{1}{\tau}(q_1^+ - q_1^c) + B_2 \tilde{q}_2^c \qquad &= \dot{g}^c \end{aligned}$$

# What's the Gain

$$\begin{aligned} M\tfrac{v^+-v^c}{\tau} + K(v^c) + B^T p^c &= f^c \\ Bv^+ \qquad\qquad &= g^+ \end{aligned}$$

Consider the original system

- The inherent equation for the pressure reads

$$BM^{-1}B^T p^c = \frac{Bv^c - Bv^+}{\tau} + BM^{-1}[f^c - K(v^c)]$$

- If we solve the algebraic up to an residual of size $\varepsilon$, then

$$\frac{Bv^c - Bv^+}{\tau} = \frac{g^c - g^+}{\tau} + \frac{\varepsilon^c - \varepsilon^+}{\tau},$$

i.e. in the pressure, the algebraic error is amplified by $\frac{1}{\tau}$.

# What's the Gain

$$BM^{-1}B^Tp^c = \frac{Bv^c - Bv^+}{\tau} + BM^{-1}[f^c - K(v^c)]$$
$$\frac{Bv^c - Bv^+}{\tau} = \frac{g^c - g^+}{\tau} + \frac{\varepsilon^c - \varepsilon^+}{\tau}$$

While for the index-1 system,

- the inherent equation for the pressure reads

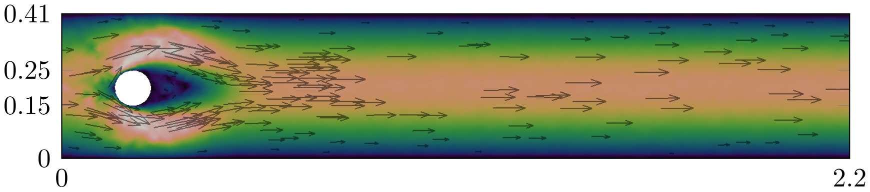
$$BM^{-1}B^Tp^c = \frac{1}{\tau}BQ\begin{bmatrix} q_1^c - q_1^+ \\ -\tau\tilde{q}_2^c \end{bmatrix} + BM^{-1}[f^c - K(q_1^c, q_2^c)],$$

- which is affected by the algebraic error via

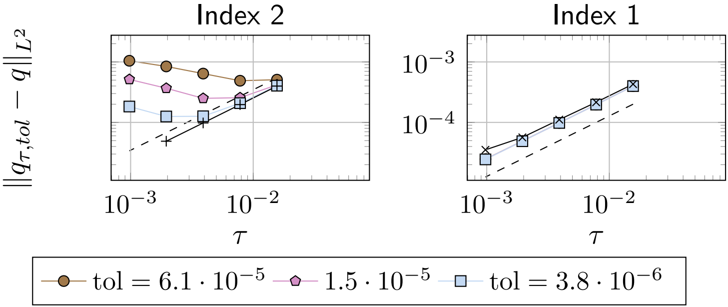
$$\frac{1}{\tau}BQ\begin{bmatrix} q_1^c - q_1^+ \\ -\tau\tilde{q}_2^c \end{bmatrix} = -\dot{g}^+ + \varepsilon^c.$$

# The Setup

- Navier-Stokes equations for the cylinder wake at $Re = 60$
- Nonuniform spatial discretization
- Crouzeix-Raviart finite elements using [FEniCS]
- about $15000$ velocity and $5000$ pressure nodes
- Uniform time-discretization
- Numerically computed reference solution
- Semi-Explicit Euler scheme
- GMRes with [KryPy] for the linear systems
- "Fair" balancing of the residuals

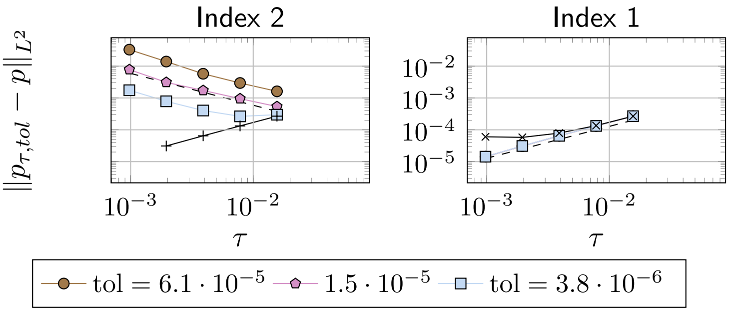


# Velocity Approximation



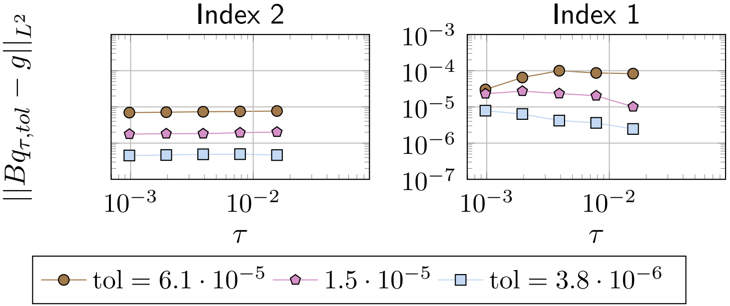

- The error in the velocity $v$ versus time step length $\tau$
- for varying tolerances $\mathrm{tol}$ in the linear solver
- the $+$ and $\times$ in the plots denote exact solves (Index-2) and very rough solves (Index-1)

# Pressure Approximation




- The error in the pressure $p$ versus time step length $\tau$
- for varying tolerances $\mathrm{tol}$ in the linear solver
- the $+$ and $\times$ in the plots denote exact solves (Index-2) and very rough solves (Index-1)

# Residual in the Divergence Constraint




- The residual in the divergence free constraint versus time step length $\tau$
- for varying tolerances $\text{tol}$ in the linear solver

# Conclusion & Outlook

Summary:

- the time integration of semi-discretized flow equations requires some index reduction (well known)
- *Minimal extension* can be formulated on the PDE level
- and numerically realized in FEM by analytical splittings of velocity ansatz spaces

(Necessary) future work:

- Efficiency for the solution of the index-1 systems
- Formulation for higher order time integration

**Thank you for your attention**

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# Literature


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# Penalty Method

A perturbation in the constraints gives the strangeness-free approximation to the NSEs

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v}_\varepsilon \\ \dot{p}_\varepsilon \end{bmatrix} - \begin{bmatrix} A & B^T \\ B & \varepsilon I \end{bmatrix} \begin{bmatrix} v_\varepsilon \\ p_\varepsilon \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

In Shen 1995 it was shown that in continous space the modelling error behaves like

# Penalty Method Backdraw

Problem: For fixed time step $\tau$ a smaller $\varepsilon$ makes the problem ill conditioned: For illustration we observe that with

$$Bv + \varepsilon p = -g \quad \text{or} \quad p = -\tfrac{1}{\varepsilon}[Bv - g]$$

the linear case reads

$$M\dot{v} - Av - B^T p = f \quad \text{or} \quad M\dot{v} - [A - \tfrac{1}{\varepsilon}B^T B]v = f - \tfrac{1}{\varepsilon}Bg$$

and the application of Backward Euler requires solves with the coefficient matrix

$$\big[M - \tau[A - \tfrac{1}{\varepsilon}B^T B]\big]$$

which is ill-conditioned if $\tau \gg \varepsilon$.

# Pressure Poisson Formulations

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{p} \end{bmatrix} - \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

Differentiation of the constraint,

$$-Bv = g \quad \text{becomes} \quad B\dot{v} = -\dot{g},$$

and the application of $BM^{-1}$ to the differential part leads to the system

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{p} \end{bmatrix} - \begin{bmatrix} A & B^T \\ BM^{-1}A & BM^{-1}B^T \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} = \begin{bmatrix} f \\ BM^{-1}f + \dot{g} \end{bmatrix},$$

which is strangeness-free, since $BM^{-1}B^T$ is invertible.

# Numerical Treatment of the PPE Formulation

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{p} \end{bmatrix} - \begin{bmatrix} A & B^T \\ BM^{-1}A & BM^{-1}B^T \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} = \begin{bmatrix} f \\ BM^{-1}f + \dot{g} \end{bmatrix}$$

- One can use standard time integration methods, e.g.
  - Backward Euler $\rightarrow$ global error is $\mathcal{O}(\tau)$ in $v$ and $p$
  - Trapezoidal rule $\rightarrow$ $\mathcal{O}(\tau^2)$
  - see Gresho/Sani 2000 for a concise discussion
- But the solution drifts off the original constraints unless one uses modified discretization schemes

# Splitting/Projection/PressureCorrection Algorithm

**Algorithm** for one time step according to

$$M\frac{u^{n+1}-u^n}{\tau} - Au^n - B^T p^{n+1} = f^{n+1} \qquad (2)$$

$$Bu^{n+1} = g^{n+1} \qquad (3)$$

1. guess a pressure gradient $B^T\tilde{p}$ and solve

$$M\frac{\tilde{u}-u^n}{\tau} - Au^n - B^T\tilde{p} = f^{n+1} \qquad (1')$$

2. Having substracted $(1)-(1')$ one finds

$$M\frac{u^{n+1}-\tilde{u}}{\tau} - B^T(p^{n+1}-\tilde{p}) = 0 \qquad (4)$$

3. Use $-Bu^{n+1} = g^{n+1}$, cf. (2), to solve (3) for $(p^{n+1}-\tilde{p})$
4. get $u^{n+1}$ from (3) and $p^{n+1} = \tilde{p} + (p^{n+1}-\tilde{p})$

# Splitting/Projection/PressureCorrection Analysis

- Straight forward implementation for nonlinear formulations
- but it amplifies the error by $\frac{1}{\tau}$ when solving for the correction
- and it also explicitly requires boundary conditions for the pressure
- It is nothing else than *Stabilization by Projection*, cf. Hairer/Wanner 2002 for convergence results

It is actually a time integration scheme for the strangeness-free system

$$\begin{bmatrix}\dot{\tilde{v}}\\0\\0\\0\end{bmatrix}-\begin{bmatrix}0 & M^{-1}A & 0 & M^{-1}B^T\\-I & I & M^{-1}B^T & 0\\0 & B & 0 & 0\\0 & BM^{-1}A & 0 & BM^{-1}B^T\end{bmatrix}\begin{bmatrix}\tilde{v}\\v\\\phi\\p\end{bmatrix}=\begin{bmatrix}f\\0\\g\\BM^{-1}f+\dot{g}\end{bmatrix}$$

# Divergence Free Bases

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot v \\ \dot p \end{bmatrix} - \begin{bmatrix} A & B^T \\ BM^{-1}A & BM^{-1}B^T \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} = \begin{bmatrix} f \\ BM^{-1}f + \dot g \end{bmatrix}$$

Find an orthogonal transformation matrix $Q$, such that $BQ = [R\ 0]$ and transform the equations to get

$$\begin{bmatrix} Q^T MQ & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} Q^T v \\ p \end{bmatrix} - \begin{bmatrix} Q^T AQ & \begin{bmatrix} R^T \\ 0 \end{bmatrix} \\ \begin{bmatrix} R & 0 \end{bmatrix} & 0 \end{bmatrix} \begin{bmatrix} Q^T v \\ p \end{bmatrix} = \begin{bmatrix} Q^T f \\ g \end{bmatrix},$$

which can be decomposed into algebraic in differential equations

- The computation of $Q$ is expensive to unfeasible
- and if it is done numerically, it may cause instabilities
- But the system is reduced $\rightarrow$ less computational effort in the time integration

# Common Solution Approaches

$$\begin{aligned} M\dot{v} + K(v) + B^T p &= f \\ Bv + \quad \epsilon p &= g \quad (*) \end{aligned}$$

(a) Splitting, projection, or pressure correction schemes
(b) Pressure penalization scheme
(c) Divergence-free methods

with common difficulties

- Need for pressure boundary conditions (a)
- Parameter dependency (b)
- Ill-conditioned resulting linear system (b)
- Instability in the pressure approximation (a-c)

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