Boundary Control of Turbulent Flow Fields? I Try Linearizations and Flow Decompositions for Controller Design

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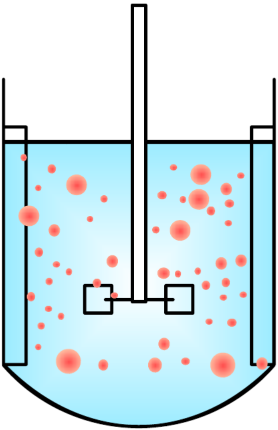
Outline of the Talk

- Introduction of the problem and the model
- Linearization of the model
- Decomposition of the flow field
- Resulting descriptor system

The Underlying Problem

- Stirring of two fluids
- Reynoldsnumber $Re \sim 30,000$
- In industrial applications:
  - continuous phase $\sim 90\%$
  - dispersed phase $\sim 10\%$
- Imagine droplets of oil swimming in water



Can we use the stirrer speed to obtain a desired droplet population in optimal time?

The Model for the Flow

The Reynolds-Averaged Navier-Stokes Equations with the $k - \varepsilon$ model:

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v + \nabla p + \frac{2}{3}\nabla k - \operatorname{div} \nu^* S = f_v$$

$$\operatorname{div} v = 0$$

$$\frac{\partial [k\varepsilon]}{\partial t} + (v \cdot \nabla)[k\varepsilon] - \operatorname{div} \Gamma_{[k\varepsilon]} \nabla [k\varepsilon] = f_{[k\varepsilon]}$$

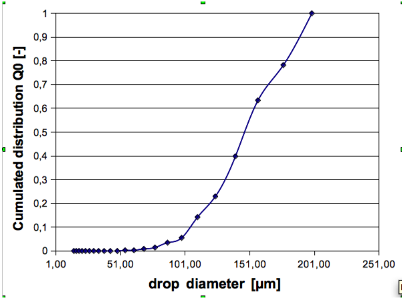
in a domain $\Omega \subset \mathbb{R}^3$ in a time interval $(0, T]$ plus completing initial and boundary conditions

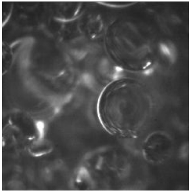
p ... pressure
v ... (mean) velocity
ε ... energy dissipation rate
k ... turbulent kinetic energy

$\nu^* = \nu^*(k, \varepsilon)$ ... turbulent kinematic viscosity
$\Gamma_{[k\varepsilon]}$ ... diffusion coefficient
$S = \nabla v^T + \nabla v$ ... stress tensor

Description and Modelling of the Droplet Population

Count the drops and measure their diameter d





to obtain the density distribution

$$n(d; x, t)$$

of the droplet population wrt to the diameter

Density Functions and Moments

The considered density distribution $n : [0, d_{\max}] \to \mathbb{R}_{\geq 0}$ is defined via

$$\int_{d_b}^{d_u} n(d)\mathrm{d}d = \text{ Number of drops with diameter } d \in [d_b, d_u).$$

Instead of computing the whole distribution we only compute some moments

$$m_k = \int_0^{d_{\max}} d^k n(d)\mathrm{d}d,$$

being aware of the illposedness of the problem

$$\{m_0, m_1, m_2, \dots\} \xrightarrow{?} n(d)$$

Use and Computation of the Moments

To describe the population one uses a characteristic diameter

$$d_{32} := \frac{\sum d_i^3}{\sum d_i^2} = \frac{m_3}{m_2}$$

with the sums over all drops and the standard deviation

$$\sigma^2 = \sum (d_i - d_{\text{mean}})^2 = \frac{m_2}{m_0} - \frac{m_1^2}{m_0^2}$$

that can be expressed in terms of the moments.
The moments $[m]$ are computed together with the flow via

$$\frac{\partial [m]}{\partial t} + (v \cdot \nabla)[m] - \operatorname{div} \Gamma_m \nabla [m] = f_m$$

The Complete System

In $\Omega \times (0, T]$ solve

$$\begin{aligned}
\frac{\partial v}{\partial t} + (v \cdot \nabla)v + \nabla p + \frac{2}{3}\nabla k - \operatorname{div} \nu^* S &= f_v \\
\frac{\partial [k\varepsilon]}{\partial t} + (v \cdot \nabla)[k\varepsilon] - \operatorname{div} \Gamma_{k\varepsilon}\nabla[k\varepsilon] &= f_{k\varepsilon} \\
\frac{\partial [m]}{\partial t} + (v \cdot \nabla)[m] - \operatorname{div} \Gamma_m \nabla[m] &= f_m \\
\operatorname{div} v &= 0
\end{aligned}$$

for given initial and boundary conditions.

Especially for the stirrer we act on the system by imposing

$$v|_{\text{stirrer}} = \omega(t)$$

Linearization

The nonlinearities

$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v + \nabla p + \frac{2}{3}\nabla k - \operatorname{div} \nu^* S = f_v$$
$$\frac{\partial [k\varepsilon]}{\partial t} + (v \cdot \nabla)[k\varepsilon] - \operatorname{div} \Gamma_{k\varepsilon} \nabla [k\varepsilon] = f_{k\varepsilon}$$
$$\frac{\partial [m]}{\partial t} + (v \cdot \nabla)[m] - \operatorname{div} \Gamma_m \nabla [m] = f_m$$
$$\operatorname{div} v = 0$$

are "fixed" by using reference trajectories of the variables:

$$v_\infty(t), k_\infty(t), \varepsilon_\infty(t) \quad \text{and} \quad [m]_\infty(t)$$

Linearized Model

which gives

$$
\begin{aligned}
\frac{\partial v}{\partial t} + (v_\infty \cdot \nabla)v + (v \cdot \nabla)v_\infty + \nabla p + \frac{2}{3}\nabla k - \operatorname{div} \nu^*_\infty S &= f_v \\
\frac{\partial [k\varepsilon]}{\partial t} + (v \cdot \nabla)[k\varepsilon]_\infty + (v_\infty \cdot \nabla)[k\varepsilon] - \operatorname{div} \Gamma_{k\varepsilon_\infty} \nabla [k\varepsilon] &= f_{k\varepsilon_\infty} \\
\frac{\partial [m]}{\partial t} + (v \cdot \nabla)[m]_\infty + (v_\infty \cdot \nabla)[m] - \operatorname{div} \Gamma_{m_\infty} \nabla [m] &= f_{m_\infty} \\
\operatorname{div} v &= 0
\end{aligned}
$$

Short Hand Form

With the abbreviations (exemplarily for the $[k\varepsilon]$) equation:

$$c_{[k\varepsilon]_\infty} \bullet := -(\bullet \cdot \nabla)[k\varepsilon]_\infty$$
$$L_{[k\varepsilon]} \bullet := \operatorname{div} \Gamma_{k\varepsilon_\infty} \nabla \bullet - (v_\infty \cdot \nabla) \bullet$$

the system writes as

$$\underbrace{\begin{bmatrix} I & & & \\ & I & & \\ & & I & \\ & & & \end{bmatrix}}_{:=E} \frac{\partial}{\partial t} \begin{bmatrix} v \\ [k\varepsilon] \\ [m] \\ p \end{bmatrix} = \underbrace{\begin{bmatrix} L_v + c_{v_\infty} & \frac{2}{3}\nabla & & -\nabla \\ c_{[k\varepsilon]_\infty} & L_{[k\varepsilon]} & & \\ c_{[m]_\infty} & & L_{[m]} & \\ \operatorname{div} & & & \end{bmatrix}}_{:=A} \begin{bmatrix} v \\ [k\varepsilon] \\ [m] \\ p \end{bmatrix} + f$$

Even Shorter Form

We write the above system as

$$\begin{bmatrix} E \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} v \\ [k\varepsilon] \\ [m] \\ p \end{bmatrix} = \begin{bmatrix} A_v \middle| A_{\mathrm{k\varepsilon mp}} \end{bmatrix} \begin{bmatrix} v \\ [k\varepsilon] \\ [m] \\ p \end{bmatrix} + f$$

with the splitting of A according to the subscripted variables. The control is still present in the boundary condition

$$v\big|_{\text{stirrer}} = \omega(t)$$

Flow Decomposition

Idea: Given stirrer speed $\omega(t)$

- Construct a divergence free flow-field $v_s(\omega)$ with

$$v_s|_{\text{stirrer}} = \omega(t)$$

- Compute the velocity in the tank as

$$v + v_s$$

- $[m], [k\varepsilon]$ remain unchanged

Decomposed System

Then the controlled system becomes

$$\left[E\right]\frac{\partial}{\partial t}\begin{bmatrix} v+v_s \\ [k\varepsilon] \\ [m] \\ p \end{bmatrix} = \left[A_v \middle| A_{\mathrm{k\varepsilon mp}}\right]\begin{bmatrix} v+v_s \\ [k\varepsilon] \\ [m] \\ p \end{bmatrix} + f$$

which is

$$\left[E\right]\frac{\partial}{\partial t}\begin{bmatrix} v \\ [k\varepsilon] \\ [m] \\ p \end{bmatrix} = \left[A_v \middle| A_{\mathrm{k\varepsilon mp}}\right]\begin{bmatrix} v \\ [k\varepsilon] \\ [m] \\ p \end{bmatrix} + \left[A_v - \begin{bmatrix} \frac{\partial}{\partial t} \\ 0 \\ 0 \\ 0 \end{bmatrix}\right] v_s + f$$

with $v\big|_{\mathrm{stirrer}} = 0$ and further boundary conditions.

Descriptor System

A spatial discretization then leads to the descriptor system:

$$\mathbf{E}(t)\frac{d}{dt}\begin{bmatrix} v \\ [k\varepsilon] \\ [m] \\ p \end{bmatrix} = \mathbf{A}(t)\begin{bmatrix} v \\ [k\varepsilon] \\ [m] \\ p \end{bmatrix} + \mathbf{B}(t)v_s + \mathbf{f}$$

with initial conditions for $v, [k\varepsilon]$ and $[m]$.

System Specific Simplifications

We use the notation

$$u := \begin{bmatrix} v_s \\ \frac{\partial}{\partial t} v_s \end{bmatrix}$$

and set wlog $\mathbf{f} = 0$.

With M denoting the mass matrix of the spatial discretization we can set

$$\mathbf{B}(t) = \begin{bmatrix} A_{vv} & -M \\ A_{[k\varepsilon],v} & 0 \\ A_{[m]v} & 0 \\ C & 0 \end{bmatrix} := \begin{bmatrix} B_v(t) \\ B_{[k\varepsilon]}(t) \\ B_{[m]}(t) \\ 0 \end{bmatrix}$$

since C is the discretized div-operator and $u = v_s$ is divergence free.

The Strangeness Becomes Visible

With C^T standing for the discretized ∇-operator the system reads

$$\begin{bmatrix} M & & & \\ & M & & \\ & & M & \\ & & & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} v \\ [k\varepsilon] \\ [m] \\ p \end{bmatrix} = \begin{bmatrix} A_{vv}(t) & \frac{2}{3}C^T & & -C^T \\ * & * & & \\ * & & * & \\ C & & & \end{bmatrix} \begin{bmatrix} v \\ [k\varepsilon] \\ [m] \\ p \end{bmatrix} + \begin{bmatrix} B_v(t) \\ B_{[k\varepsilon]}(t) \\ B_{[m]}(t) \\ 0 \end{bmatrix} u$$

This is a differential-algebraic equation system.

The Strangeness Becomes Visible

The strangeness is hidden in the rows and lines connected with p

$$
\begin{bmatrix} M & & & \\ & M & & \\ & & M & \\ & & & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} v \\ [k\varepsilon] \\ [m] \\ p \end{bmatrix} =
$$

$$
\begin{bmatrix} A_{vv}(t) & \frac{2}{3} C^T & & -C^T \\ * & * & & \\ * & & * & \\ C & & & \end{bmatrix} \begin{bmatrix} v \\ [k\varepsilon] \\ [m] \\ p \end{bmatrix} + \begin{bmatrix} B_v(t) \\ B_{[k\varepsilon]}(t) \\ B_{[m]}(t) \\ 0 \end{bmatrix} u
$$

and can be removed by hand. . .

Removing the Strangeness

Thereto consider the first and the last line given by

$$M\dot{v} = A_{vv}(t)v + \frac{2}{3}C^T k - C^T p + B_v(t)u$$
$$0 = Cv.$$

Rearrange the first and differentiate the last:

$$\dot{v} + M^{-1}C^T p = M^{-1}(A_{vv}(t)v + \frac{2}{3}C^T k + B_v(t)u)$$
$$0 = C\dot{v}$$

Apply C to the first and use the second to get

$$CM^{-1}C^T p = CM^{-1}(A_{vv}(t)v + \frac{2}{3}C^T k + B_v(t)u)$$

or, since $CM^{-1}C^T$ is invertible for reasonable discretizations,

$$C^T p = \underbrace{C^T(CM^{-1}C^T)^{-1}CM^{-1}}_{:=\mathscr{P}}(A_{vv}(t)v + \frac{2}{3}C^T k + B_v(t)u)$$

The Strangeness-free System

Thus, replacing $C^T p$ in the descriptor system using

$$C^T p = \underbrace{C^T (CM^{-1} C^T)^{-1} CM^{-1}}_{:=\mathscr{P}} (A_{vv}(t)v + \frac{2}{3} C^T k + B_v(t)u)$$

one obtains the equivalent ODE system,

$$\begin{bmatrix} M & & \\ & M & \\ & & M \end{bmatrix} \frac{d}{dt} \begin{bmatrix} v \\ [k\varepsilon] \\ [m] \end{bmatrix} =$$

$$\begin{bmatrix} (I - \mathscr{P})A_{vv}(t) & 0 & \\ * & * & \\ * & & * \end{bmatrix} \begin{bmatrix} v \\ [k\varepsilon] \\ [m] \end{bmatrix} + \begin{bmatrix} (I - \mathscr{P})B_v(t) \\ B_{[k\varepsilon]}(t) \\ B_{[m]}(t) \end{bmatrix} u$$

for which one can apply e.g. the LQR-theory for controller design

Conclusion and Outlook

Conclusion

- Linearization AND flow decomposition transform the original system into a descriptor system with variable coefficients
- For these systems there are some theoretical results avaliable concerning controller design
- A possible way is to find a strangeness-free formulation and use standard theory

Upcoming Tasks

- Analyse the original and the descriptor system for controllability
- Design the controller
- The controller will give the $v_s(t)$, but for the simulation we need $\omega(t)$

Last Slide

- Thank you for your attention
- I am very interested in your advice and suggestions