Numerical Methods for PDAE Constraint Optimal Control

Jan Heiland*, Volker Mehrmann*

*TU Berlin, Department of Mathematics

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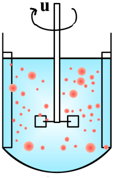

$$\overset{\mathbf{u}}{\longleftrightarrow} \qquad \$$$

In mathematical terms :

$$\begin{aligned} \frac{\partial v}{\partial t} - f(v,p,u) &= 0 \\ g(v) &= 0 \end{aligned} \qquad \longleftrightarrow \qquad \mathcal{J}(v,p,u) \to \min$$

$$\frac{\partial v}{\partial t} - f(v, p, u) = 0$$
$$g(v) = 0$$

formulated in time and space

Approximate the state via a finite dimensional linear system

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} v \\ p \end{bmatrix} - \begin{bmatrix} D(t) & J_2^T \\ J_1 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} - \begin{bmatrix} B_1(t) \\ 0 \end{bmatrix} u = 0, \quad v(0) = v^0$$

with

- $M \in \mathbb{R}^{n_1,n_1}$ invertible
- $J_1 M^{-1} J_2^T \in \mathbb{R}^{n_2,n_2}$ invertible ($\rightarrow$ index 2)

$$\mathcal{J}(v,p,u) \to \min$$

and the cost functional via a quadratic form

$$\mathcal{J}(v,p,u) = \frac{1}{2}\left\{ \begin{bmatrix} v(T) \\ p(T) \end{bmatrix}^T \begin{bmatrix} V_1 v(T) \\ V_2 p(T) \end{bmatrix} + \int_0^T \begin{bmatrix} v \\ p \end{bmatrix}^T \begin{bmatrix} W_1 v \\ W_2 p \end{bmatrix} + u^T R u \, \mathrm{d}t \right\}$$

with

- V_1, V_2, W_1, W_2 symmetric positive semidefinite (spsd)
- R spd

Given the optimal control problem

$$v(T)^T V_1 v(T) + \int_0^T v^T W_1 v + u^T R u \, \mathrm{d}t \to \min$$

$$\text{s.t.} \quad \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} v \\ p \end{bmatrix} - \begin{bmatrix} D & J_1^T \\ J_2 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} - \begin{bmatrix} B \\ 0 \end{bmatrix} u = 0, \quad v(0) = v^0.$$

Assume M and $J_2 M^{-1} J_1^T$ are invertible, $J_2 v^0 = 0$, W_1, V_1 spsd, $J_1 M^{-T} V_1 = 0$ and R spd.

Then any optimal u is given by the feedback law $u = R^{-1} B^T X M v$, where $X \in \mathbb{R}^{n_1, n_1}$ is the symmetric solution to

$$\begin{aligned} M^T \dot{X} M + D^T X M + M^T X D + M^T X R_B X M - & \\ -W_1 + J_2^T Y M + M^T Y^T J_2 &= 0 \\ J_1 X M &= 0 \\ M^T X(T) M &= -V_1. \end{aligned}$$

- Kunkel, Mehrmann, several papers on optimal control for DAEs (e.g. 1997, 2008)
- Kurina, März: *Feedback Solutions of Optimal Control Problems with DAE Constraints* (2007)
- Backes: *Optimale Steuerung der linearen DAE im Fall Index 2* (2006)
- Bänsch, Benner: *Stabilization of Incompressible Flow by Riccati-based Feedback* (2010)
  based on theoretical work by J.P. Raymond

$$v^T(T)V_1v(T) + \int_0^T v^TW_1v + u^TRu \,\mathrm{d}t$$

The optimality system is given by the Euler-Lagrange equations

$$\begin{aligned}
\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} v \\ p \end{bmatrix} - \begin{bmatrix} D & J_1^T \\ J_2 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} - \begin{bmatrix} B \\ 0 \end{bmatrix} u &= 0, \\
v(0) &= v^0, \\
\begin{bmatrix} M^T & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} D^T & J_2^T \\ J_1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} - \begin{bmatrix} W_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} &= 0, \\
M^T\lambda_1(T) &= -V_1v(T),
\end{aligned}$$

and

$$-B^T\lambda_1 + Ru = 0.$$

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} v \\ p \end{bmatrix} - \begin{bmatrix} D & J_1^T \\ J_2 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} - \begin{bmatrix} B \\ 0 \end{bmatrix} u = 0 \quad \text{(DAE)}$$

$$\begin{bmatrix} M^T & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} D^T & J_2^T \\ J_1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} - \begin{bmatrix} W_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} = 0 \quad \text{(adDAE)}$$

$$-B^T \lambda_1 + Ru = 0$$

If (DAE) and (adDAE) are such that $\dot{u}$ and $\dot{v}$ do not appear in the solution of (DAE) and (adDAE), respectively, then

$$u^* \text{ is optimal} \Leftrightarrow \begin{cases} \text{there is } (v^*, p^*, \lambda_1^*, \lambda_2^*, u^*) \\ \text{that satisfies the} \\ \text{Euler-Lagrange equations.} \end{cases}$$

cf. Backes (2006)

This means, that if we find a solution $(v, p, \lambda_1, \lambda_2)$ of the reduced optimality system

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} v \\ p \end{bmatrix} - \begin{bmatrix} D & J_1^T \\ J_2 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} - \begin{bmatrix} BR^{-1}B^T & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} M^T & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} D^T & J_2^T \\ J_1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} - \begin{bmatrix} W_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} = 0,$$

$$v(0) = v^0 \quad \text{and} \quad M^T \lambda_1(T) = -V_1 v(T)$$

then $u = R^{-1}B^T\lambda_1$ is an optimal input.

We make the ansatz:

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} X & Y^T \\ Y & 0 \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix}$$

$$\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} = \begin{bmatrix}X & Y^T\\ Y & 0\end{bmatrix}\begin{bmatrix}M & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}v\\ p\end{bmatrix}$$

Then

$$\begin{bmatrix}M^T & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot\lambda_1\\ \dot\lambda_2\end{bmatrix} = \begin{bmatrix}M^TX & M^TY^T\\ 0 & 0\end{bmatrix}\begin{bmatrix}M & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot v\\ \dot p\end{bmatrix} + \begin{bmatrix}M^T\dot XM & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}v\\ p\end{bmatrix}$$

or, having employed (DAE) and (adDAE),

$$\begin{bmatrix}M^T\dot XM + D^TXM + M^TXD + M^TXR_BXM - \\ -W_1 + J_2^TYM + M^TY^TJ_2 & M^TXJ_1^T\\ J_1XM & 0\end{bmatrix} = 0,$$

$$M^TX(T)M = -V_1.$$

Assume $M = I$ to write in short:

$$\dot{X} + D^T X + XD + XR_B X - W_1 + J_2^T Y + Y^T J_2 = 0 \quad (*)$$
$$J_1 X = 0 \quad \text{and} \quad XJ_1^T = 0$$

We use the projector $\mathcal{P} := I - J_2^T (J_1 J_2^T)^{-1} J_1$, with

$$J_1 \mathcal{P} = 0 \quad \text{and} \quad \mathcal{P} J_2^T = 0.$$

With this we obtain for $X = [\mathcal{P} + [I - \mathcal{P}]]\ X\ [[I - \mathcal{P}^T] + \mathcal{P}^T]$

- $J_1 X = 0$ and $XJ_1^T = 0 \quad \Rightarrow \quad X = \mathcal{P} X \mathcal{P}^T$
- $\mathcal{P} X \mathcal{P}^T$ is uniquely defined by the standard differential Riccati equation obtained from $\mathcal{P} \to (*) \leftarrow \mathcal{P}^T$

$$\dot{X} + D^T X + XD + XR_B X - W_1 + J_2^T Y + Y^T J_2 = 0 \quad (*)$$
$$J_1 X = 0 \quad \text{and} \quad XJ_1^T = 0$$

- $[I - \mathcal{P}] \to (*) \leftarrow \mathcal{P}^T$ uniquely defines $Y\mathcal{P}^T$
- $[I - \mathcal{P}] \to (*) \leftarrow [I - \mathcal{P}^T]$ leads to

$$[I - \mathcal{P}]Y^T J_2 + J_2^T Y[I - \mathcal{P}^T] = [I - \mathcal{P}]W_1[I - \mathcal{P}^T],$$

which defines $Y[I - \mathcal{P}^T]$ up to an additive component ZJ_2.

Thus the considered differential algebraic matrix Riccati equation has a solution and the proposed decoupling gives the feedback-law for the optimal input.

What has been presented

- Linear time-varying DAEs as a basic model for optimal control of PDAEs
- For semi-explicit DAEs of index 2 the Euler-Lagrange give sufficient conditions
- Riccati-Ansatz leads to a feedback solution for the optimal control

Upcoming

- Similar results for the ∞-dimensional system
- Numerical solution strategies for the Riccati DAEs
- Application to nonlinear problems