Decoupling and Optimal Control of Semi-explicit Semi-linear PDAEs of Index 2

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Optimal Control of ADAEs

We consider the optimal control problem

$$\min_u \mathcal{J}(v,u) := \mathcal{M}(x(T)) + \int_0^T \mathcal{K}(t, v(t), u(t) \, \mathrm{d}t$$

for function-valued v and u satisfying the abstract DAE

$$\begin{aligned} \dot{v} - A(v) - J_1' p &= f \\ J_2 v \qquad &= g \end{aligned} \tag{ADAE}$$

almost everywhere on $(0, T]$.

Optimality System

$$\mathcal{J}(v,u) \to \min, \text{ s.t.}$$
$$\dot{v} - A(v) - J_1' p - f = 0 \text{ and}$$
$$J_2 v - g = 0 \text{ (ADAE)}$$

We want to find an optimal solution via the optimality conditions

$$\begin{bmatrix} \text{(ADAE)} \\ \text{(adADAE)} \\ \text{Gradient Cond.} \end{bmatrix} \begin{bmatrix} (v,p) \\ (\lambda_v, \lambda_p) \\ u \end{bmatrix} = 0,$$

where

- (adADAE) is the adjoint to (ADAE) wrt. $\mathcal{J}$
- and (λ_v, λ_p) is the adjoint state.

Assumption 1

In the neighborhood of an optimal pair (v^*, u^*), let

a.) the involved operators be two times Fréchet differentiable,

b.) the linearized (ADAE) have a unique solution for any u,

c.) the (adADAE) have a solution,

d.) there be a constant c such that

$$-\langle h_v, \lambda_v A'_{;vv}(v^*)h_v \rangle + \mathcal{J}_{;(v,u)^2}(v^*, u^*)[h_v, h_u]^2 \geq c\|h_u\|^2$$

for all (h_v, h_u) that solve the linearized (ADAE).

Theorem

If Assumption 1 holds, then (v^*, u^*) is a locally unique solution to the optimal control problem. This solution can be approached by Newton's method for the nonlinear optimality system.

Agenda

In this talk, I will address the parts

b.) solvability of the (ADAE)

c.) and the (adADAE)

of Assumption 1 by introducing a splitting of ADAEs.

This splitting separates the algebraic and differential variables, such that one can read off consistency and smoothness conditions and use standard abstract theory for existence of solutions to evolution equations.

Functional Analysis Setting

- Banach space $V \subset H$ Hilbert space (dense and continuously embedded)
- Riesz isomorphism $j' \colon H \to H'$ identifies H with its dual H'
- Then, $V \subset H \cong H' \subset V'$

We look for $v \in \big((0,T) \to V\big)$ with $\dot v(t) \in V'$ and for $p \in \big((0,T) \to Q\big)$ that satisfy

$$
\begin{aligned}
\dot v(t) - A(v(t)) - J_1' p(t) &= f(t) \quad \text{in } V', \text{ a.e. in } (0,T),\\
-J_2 v(t) \qquad\qquad &= g(t) \quad \text{in } Q', \text{ a.e. in } (0,T).
\end{aligned}
$$

with $V \subset H \subset V'$ as defined above, and with a Hilbert space Q.

Abstract ADAE

$$
\begin{aligned}
v(t) \in V,\ \dot v(t) \in V',\ p(t) \in Q,& \\
\dot v(t) - A(v(t)) - J_1' p(t) = f(t) &\quad \text{in } V' \\
-J_2 v(t) \qquad\qquad\qquad = g(t) &\quad \text{in } Q'
\end{aligned}
$$

With operators

- $A\colon V \to V'$
- $J_1' : Q \to V'$
- $J_2 : V \to Q'$

Major obstacle for decoupling:
The equations are posed in a Banach space $V' \supsetneq V$

$\rightarrow$ A projection does not necessarily split the space

$\rightarrow$ Solution space is different from "equation space"

Strategy

We proceed as follows:

- Given a smooth f and smoothing operators,
- the equation is posed in H' – rather than in V' –
- which is a Hilbert space.
- There we can define projectors and decouple the equations.
- $v \in V \subset H$ can be decoupled with the dual projections.

[Well posedness] and [Smoothness] Assumption

$$\begin{aligned} \dot{v}(t) - A(v(t)) - J_1' p(t) &= f(t) \quad \text{in } V' \\ -J_2 v(t) &= g(t) \quad \text{in } Q' \end{aligned}$$

Assumption [WP]

(a) $J_1', J_2' \in \mathcal{L}(Q, H')$ are homeomorphisms onto their range and

(b) $j(\operatorname{im} J_1') \cap \ker J_2 = \{0\}$,

where $j\colon H' \to H$ is the Riesz isomorphism.

Assumption [S]

For more regular data $f(t) \in H'$ (rather than in V') any corresponding solution (v, p) of the abstract DAE invokes $A(v(t)) \in H'$.

Splitting of the ADAE

Theorem

Under Assumptions [WP], we have

$$H' = j(\ker J_2) \oplus \operatorname{im} J_1' =: H'_{df} \oplus H'_g,$$

defining a projector $\mathcal{P}_{H'_{df}}$. If, in addition [S] holds, then, for f and g sufficiently smooth, any v with values in V solving the ADAE

$$\begin{aligned} \dot v(t) - A(v(t)) - J_1' p(t) &= f(t) \quad \text{in } V', \\ -J_2 v(t) \qquad\qquad\qquad &= g(t) \quad \text{in } Q', \end{aligned}$$

a.e. in $(0, T)$, can be written as $v = v_{df} + v_g$, where

$$\begin{aligned} j' v_g(t) &= -J_1' S^{-1} g(t) \text{ in } H'_g \\ \dot v_{df}(t) - \mathcal{P}_{H'_{df}} A\big(v_{df}(t) - jJ_1 S^{-1} g(t)\big) &= \mathcal{P}_{H'_{df}} f(t) \text{ in } H'_{df}. \end{aligned}$$

What Else?

Further issues:

- Regularity of $\big(t \mapsto v(t), p(t)\big)$
- Initial conditions and consistency
- Spatial discretizations

Thanks to Volker Mehrmann and

thank you for your attention.

For suggestions and questions please contact me

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