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Efficient Numerical Approximation of General Flow Stabilization Problems with Boundary Actuation

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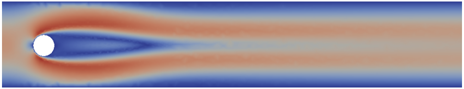
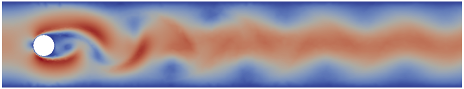
MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

Introduction

Outline of the Talk



- A general numerical tool for
- optimal control of flows
- by means of a back end that
  - → accounts for general structures
    - → copes with high dimensions
      - → covers typical applications



Optimal Control of Flows

The General Structure

We consider model equations for approximations v and p to the velocity and pressure of a flow

$$\begin{aligned} M\dot{v} - Av - J^T p &= Bu, \\ Jv \qquad\quad &= 0, \\ y &= Cv, \end{aligned}$$

- M ... mass matrix, invertible
- A ... convection and diffusion operator
- J ... discrete divergence
- B, C ... input and output operators

And remark that

- the dynamics are constrained to the kernel of J,
- the dimension of the state spaces of v and p can be very large.

Optimal Control of Flows

Constrained Riccati Equations

The constraints and structure of the dynamical equations are reflected in the Riccati equations we need solve for the optimization.

Find $X \in \mathbb{R}^{n_v,n_v}$ symmetric negative semi-definite that solves

$$A^T XM + M^T XA - M^T XBB^T XM + CC^T + MYJ^T + JY^T M^T = 0,$$

with constraints $JXM^T = 0$ and $MXJ^T = 0$, for a suitable $Y \in \mathbb{R}^{n_v,n_p}$.

Optimal Control of Flows

Constrained Riccati Equations

At the core we need to solve

$$A^T XM + M^T XA - M^T XBB^T XM + CC^T + MYJ^T + JY^T M^T = 0,$$

- Nonlinear in $X \in \mathbb{R}^{n_v, n_v}$
  - ➛ Newton iterations are well defined
- High dimensional: n_v is typically large
  - ➛ Low-rank ADI iterations are a solution
  - ➛ We compute but a skinny factor $Z \in \mathbb{R}^{n_v, k}$ so that $ZZ^T \approx -X$

Optimal Control of Flows

Efficient Solvers for Constrained Riccati Equations

At the core we need to solve linear systems of the form

$$\begin{bmatrix} A - \sigma_i M & J^T \\ J & 0 \end{bmatrix} \begin{bmatrix} z_i \\ * \end{bmatrix} = r.h.s.$$

for column blocks z_i of the factor Z.

This, we can do efficiently by means of

- Optimized ADI parameters $\sigma_i \in \mathbb{C}_-$ (the shifts)
- Update formulas for the residuals
- Performant block preconditioners for shifted saddle point problems in flow simulations

Related Work

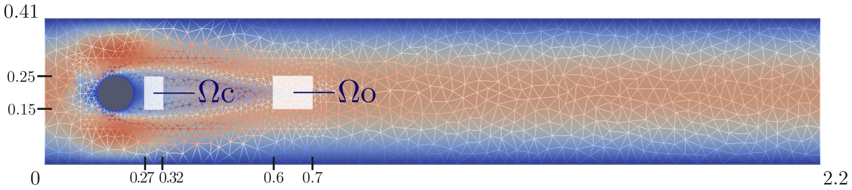
Raymond. Feedback boundary stabilization of the two-dimensional Navier-Stokes equations
SIAM J. Cont. Opt. 45:790–828, 2006.

Heinkenschloss, Sorensen, Sun. Balanced truncation model reduction for a class of descriptor systems with applications to the Oseen equations.
SIAM J. Sci. Comput., 30:1038–1063, 2008.

Bänsch, Benner, Saak, Weichelt. Riccati-based boundary feedback stabilization of incompressible Navier-Stokes flow.
Preprint SPP1253-154, DFG-SPP1253, 2013.

Application Example

Simulation Setup



- 2D cylinder wake
- Navier-Stokes Equations
- *Re* = 200
- *Taylor-Hood* finite elements in FEniCS
- 20000 velocity nodes

- Distributed control and observation with 6 degrees of freedom each
- LQGBT-reduced order observer and controller of state dimension $n_k = 27$
- Target: stabilization of the steady-state solution

Application Example

Simulation Results

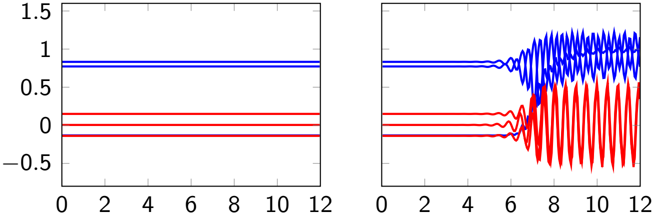
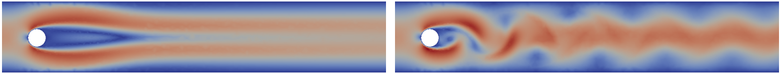


Figure : Measured output y versus time $t \in [0, 12]$ of the closed loop system (left), compared to the response of the uncontrolled system (right).
Below, a snapshot of the magnitude of the velocity solutions at $t = 12$.



Time-varying Dirichlet BC

The Basic Assumption

We consider model equations for approximations v and p to the velocity and pressure of a flow

$$\begin{aligned} M\dot{v} - Av - J^T p &= Bu, \\ Jv \qquad\quad &= 0, \\ y &= Cv, \end{aligned}$$

- M, A, J ... coefficient matrices
- B, C ... input and output operators

And remark that

→ for boundary control, the appearance of the input as Bu in the dynamical part is by no means immediate

Time-varying Dirichlet BC

Definition of B

The input operator is defined by the incorporation of the time dependent Dirichlet boundary conditions in the spatial discretization.

Consider the convection diffusion equation:

$$\begin{aligned} \dot{v}(t) + \beta \cdot \nabla v(t) - \Delta v(t) &= 0, \\ v|_\Gamma(t) &= u(t), \end{aligned}$$

for a scalar quantity v with a wind β in the domain Ω.
Here, u is the function that prescribes the values of v at the boundary Γ.

Time-varying Dirichlet BC

Direct Assignment

Take a standard FE space with a nodal basis and decompose it into subspaces

$$V = V_I \oplus V_\Gamma = \operatorname{span}\{\phi_I^k\}_{k=1}^{n_I} \oplus \operatorname{span}\{\phi_\Gamma^k\}_{k=1}^{n_\Gamma}$$

of inner and boundary nodes. Then, for the ansatz

$$v(t) = \sum_{k=1,\cdots,n_I} v_I^k(t)\phi_I^k + \sum_{k=1,\cdots,n_\Gamma} v_\Gamma^k(t)\phi_\Gamma^k \leftrightarrow \begin{bmatrix} v_I \\ v_\Gamma \end{bmatrix}(t)$$

and having tested the equations against $\phi \in V_I$, we obtain

$$\begin{bmatrix} M_{II} & M_{I\Gamma} \end{bmatrix} \begin{bmatrix} \dot{v}_I \\ \dot{v}_\Gamma \end{bmatrix} = \begin{bmatrix} A_{II} & A_{I\Gamma} \end{bmatrix} \begin{bmatrix} v_I \\ v_\Gamma \end{bmatrix}$$

$$v_\Gamma = u$$

Time-varying Dirichlet BC

Direct Assignment and Alternatives

$$\begin{bmatrix} M_{II} & M_{I\Gamma} \end{bmatrix} \begin{bmatrix} \dot{v}_I \\ \dot{v}_\Gamma \end{bmatrix} = \begin{bmatrix} A_{II} & A_{I\Gamma} \end{bmatrix} \begin{bmatrix} v_I \\ v_\Gamma \end{bmatrix}$$

$$v_\Gamma = u$$

A direct elimination of the boundary nodes, thus, leads to

$$M_{II} \dot{v}_I = A_{II} v_I + A_{I\Gamma} u - M_{I\Gamma} \dot{u}$$

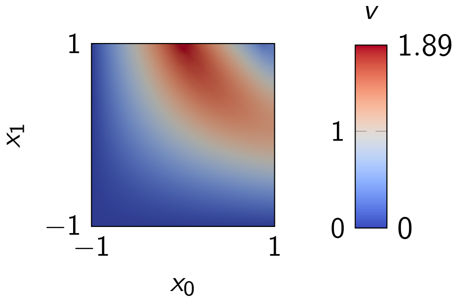
which is not like $M\dot{v} = Av + Bu$.

Possible bypasses:

- lift – lifting of the boundary conditions
- proj – incorporation via Lagrange multiplier and projections
- pero – relaxation via approximating Robin conditions

Time-varying Dirichlet BC

Convection-Diffusion Example



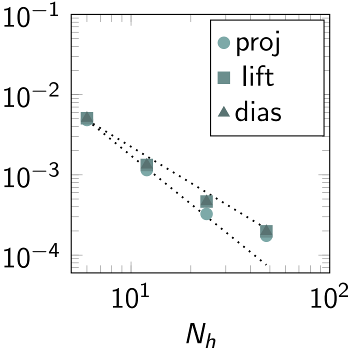
- 2D convection-diffusion, no source term
- Dirichlet condition at upper boarder

$$v\big|_{\Gamma_0} = \frac{1}{2}(\sin(\pi x_0 + \frac{\pi}{2}) + 1)(\cos(t + \pi) + 1)$$

- FE discretization using FEniCS on a uniform triangulation

Time-varying Dirichlet BC

Convergence for Boundary Forcing



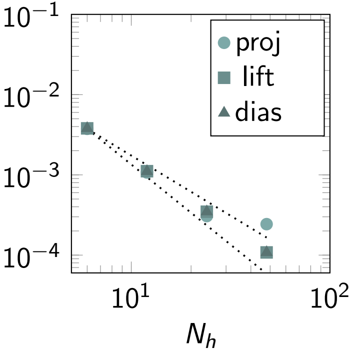


Figure : Discrete L^2-time-space error for varying space discretizations N_h, for a fixed time discretization and for linear (left) and quadratic (right) shape functions. The dotted lines indicate the slope of a convergence of order 2 or 1.5.

Time-varying Dirichlet BC

Performance of Penalization Schemes

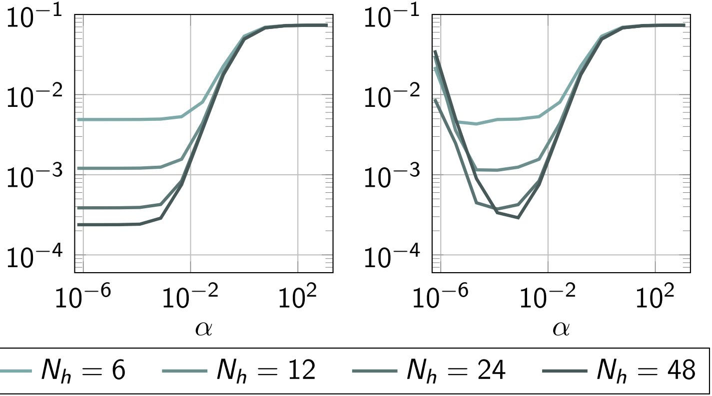


Figure : Discrete L^2-time-space error for the Robin relaxation versus the penalization parameter α error for varying space discretizations N_h for direct solves (left) and for iterative solves (right) up to a certain tolerance of the algebraic equations.

Summary
and Conclusion

- Efficient solution of large scale constrained Riccati equations
  - ➙ General tool for control of flows
- This is a system-theoretical approach
  - ➙ Requires a control of distributed type
  - ✗ which is not immediate for Dirichlet boundary control
- There are various numerical approaches for time-dependent Dirichlet conditions
  - ✗ Implementation differs from methods for steady state problems
  - ✗ Penalization methods come with extra restrictions on accuracy

Thank you for your attention!

`janheiland.de`

`github.com/highlando`

Numerical Example

Convergence with Manufactured Solution

$N_h \backslash N_\tau$	30	60	120
6	1.0000	0.9991	0.9989
12	0.2425	0.2420	0.2418
24	0.0610	0.0605	0.0604
48	0.0159	0.0153	0.0151

$N_h \backslash N_\tau$	60	120	240	480
6	0.9983	0.9979	0.9978	0.9978
12	0.1127	0.1123	0.1123	0.1123
24	0.0149	0.0128	0.0127	0.0126
48	0.0079	0.0024	0.0016	0.0015

Table : The normalized approximation error for a convection-diffusion problem with a known solution for varying space N_h and time discretizations N_τ and for ansatz and test functions of polynomial degree 1 and 2.