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The Pressure Manifold in the Unsteady Navier-Stokes Equation and in Semi-Discretizations

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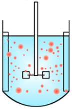
The Big Picture



abstract DAE

decoupled ADAE

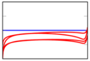
semi-discretization

convergence

finite-dimensional DAE

decoupled DAE

transformation



Outtakes

General Assumptions:

- $\rightarrow$ The right hand sides are smooth
- $\rightarrow$ The projections used are well defined
- $\rightarrow$ There are consistent initial values

Finite-dimensional Setup

$$n_v,\, n_p \in \mathbb{N},\; n_v > n_p$$
$$A_k \colon \mathbb{R}^{n_v} \to \mathbb{R}^{n_v},\; M_k \in \mathbb{R}^{n_v, n_v},\; M_k \succ 0$$
$$J_k \in \mathbb{R}^{n_v, n_p},\; J_k M^{-1} J_k^{\mathsf{T}} \text{ invertible}$$

Given $T > 0$, $f_k \colon [0, T] \to \mathbb{R}^{n_v}$, and $g_k \colon [0, T] \to \mathbb{R}^{n_p}$,

find $v_k \colon [0, T] \to \mathbb{R}^{n_v}$ and $p_k \colon [0, T] \to \mathbb{R}^{n_p}$ such that

$$M_k \dot{v}_k - A_k(v_k) - J_k^{\mathsf{T}} p_k = f_k, \tag{1a}$$
$$J_k v_k = g_k \tag{1b}$$

in $(0, T)$.

Basic observations for the decoupling

$$\begin{aligned} M_k \dot{v}_k - A_k(v_k) - J_k^\mathsf{T} p_k &= f_k \\ J_k v_k &= g_k \end{aligned}$$

We observe that

1. The state space for v and the space where the differential equation is posed can be splitted as follows:

$$\mathbb{R}^{n_v} = \ker J_k \oplus \operatorname{im} M_k^{-1} J_k^\mathsf{T}$$

2. Differentiating the algebraic constraints gives

$$J_k \dot{v}_k = \dot{g}_k$$

Decoupling

$$P := I - M_k^{-1} J_k^{\mathsf{T}} S^{-1} J_k$$
$$S := J_k M_k^{-1} J_k^{\mathsf{T}}$$

Theorem (1)

Each solution (v_k, p_k) *of the state equations can be represented as* $(v_{P_k} + v_{Q_k}, p_k)$, *where*

$$v_{Q_k} = -M_k^{-1} J_k^T S^{-1} g_k,$$
$$p_k = -S^{-1} J_k M_k^{-1} [A_k(v_{Q_k} + v_{P_k}) + f_k - M_k \dot{v}_{Q_k}],$$

and where $v_{P_k} := P v_k$ *satisfies the ODE*

$$\dot{v}_{P_k} - P M_k^{-1} A_k (v_{Q_k} + v_{P_k}) = P M_k^{-1} f_k.$$

Implications of Theorem (1)

- The DAE is decoupled
  $\rightarrow$ algebraic and differential parts are separated

- If J_k is the discrete divergence, then $S = J_k M_k^{-1} J_k^{\mathsf{T}}$ is the discrete *Laplacian* and

$$\rightarrow \quad Sp_k = -J_k M_k^{-1}[A_k(v_{Q_k} + v_{P_k}) + f_k - M_k \dot{v}_{Q_k}]$$

  is the *Pressure Poisson Equation* (PPE)

The (PPE) is used in common time integration schemes for Navier-Stokes equations...

1. Is there a PPE on the ADAE level?
2. Does the discrete PPE converge to the continuous PPE?

The Abstract Setup

Gelfand triple: $V \hookrightarrow H \cong H' \hookrightarrow V'$
Hilbert space: Q_H
$\mathcal{A}\colon V \to V'$
$\mathcal{J}\colon V \to Q_H'$

We look for

- $v\colon (0, T) \to V$, with $\dot{v}(t) \in V'$,
- and for $p\colon (0, T) \to Q_H$,

that satisfy

$$\begin{aligned} \dot{v} - \mathcal{A}(v) - \mathcal{J}'p &= f \quad \text{in } (0, T) \times V', \\ \mathcal{J}v &= 0 \quad \text{in } (0, T) \times Q_H'. \end{aligned}$$

Abstract Equations

$$\begin{aligned} \dot v - \mathcal{A}v - \mathcal{J}'p = f \quad &\text{in } (0,T) \times V', \\ \mathcal{J}v = 0 \quad &\text{in } (0,T) \times Q_H'. \end{aligned}$$

I will address three particular issues:

1. Why the finite-dimensional approach fails
2. What additional regularity helps with
3. Sufficient conditions for a decoupling

Decoupling - What Goes Wrong

- We want to decouple

$$\begin{aligned} \dot{v} - \mathcal{A}v - \mathcal{J}'p = f \quad &\text{in } V', \\ \mathcal{J}v = 0 \quad &\text{in } Q_H'. \end{aligned}$$

- The finite dimension's approach fail, because of one major reason:

$$V \subsetneq V'.$$

$\rightarrow \ \mathcal{J}\dot{v} = \ ? \quad$ (not defined yet)
$\rightarrow \ V' = (\ker \mathcal{J})' \oplus ?$

Two Problems, One Solution: Regularity

$$V \hookrightarrow H \cong H' \hookrightarrow V'$$
$$Q_H \cong Q_H'$$
$$\mathcal{J} \colon V \to Q_H'$$

To the setup

$$\begin{array}{ccccccc} V & \hookrightarrow & H & \cong & H' & \hookrightarrow & V' \\ & \searrow^{\mathcal{J}} & & & & \nearrow^{\mathcal{J}'} & \\ & & Q_H & \cong & Q_H' & & \end{array}$$

we introduce a Banach space $Q \hookrightarrow Q_H$, such that $\mathcal{J}'(Q) \subset H'$.

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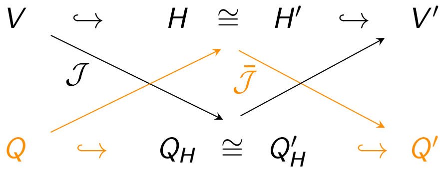

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Smoothness Assumptions

$$\mathcal{J}' \colon Q \hookrightarrow Q_H \quad \to \quad H' \hookrightarrow V'$$

Assumption (S1)

We assume that the shift of $\mathcal{J}' \colon Q_H \to V'$ "to the left":

$$\bar{\mathcal{J}}' \colon Q \to H'$$

has a left inverse.

Assumption (S2)

For more regular data $f(t) \in H'$ (rather than in V'),
any corresponding solution (v, p) is such that $\dot{v}(t)$ and $\bar{\mathcal{J}}' p(t)$
is in H', rather than in V'.

Decoupling of the ADAE

Theorem (2)

Consider $V \hookrightarrow H$, $Q \hookrightarrow Q_H$, and $\mathcal{J}$ and $\bar{\mathcal{J}}$, as defined above. If $f \in L^2(0,T;H')$, if $\mathcal{J}$ has a right inverse, and if Assumptions (S1) and (S2) hold, then any solution (v,p) to the ADAE satisfies

$$\dot{v}(t) - \mathcal{P}\mathcal{A}v(t) = \mathcal{P}f(t) \quad \text{in } j(\ker \bar{\mathcal{J}})$$

and

$$-\bar{\mathcal{J}}j\mathcal{A}v(t) - \bar{\mathcal{J}}j\bar{\mathcal{J}}'p(t) = \bar{\mathcal{J}}jf(t) \quad \text{in } Q',$$

on $(0,T)$, a.e.. Here,

- *$j\colon H' \to H$ is the Riesz-isomorphism and*
- *the projector $\mathcal{P}\colon H' \to H'$ splits $H' = j'(\ker \bar{\mathcal{J}}) \oplus \operatorname{im} \bar{\mathcal{J}}'$.*

Implications of Theorem (2)

- The ADAE can be decoupled
  - → algebraic and differential parts are separated
- Assumptions (S1) and (S2) are fulfilled for standard weak formulations of Navier-Stokes equations on regular domains
  - → Theorem 2 defines a *Pressure Poisson Equation* in infinite dimensions
- The decoupling is done analoguously to the semi-discrete approximations
  - → convergence of Galerkin schemes can be shown

Convergence of Semi-Discretizations

Consider a mixed Galerkin scheme

$$\{V_k\}_{k\in\mathbb{N}} \to V \quad \text{and} \quad \{Q_k\}_{k\in\mathbb{N}} \to Q.$$

When do the solutions p_k and v_{P_k} with

$$p_k(t) \in Q_k \quad \text{and} \quad v_{P_k}(t) \in V_k$$

of the discrete decoupled equations

$$\begin{aligned} \dot{v}_{P_k} &= PM_k^{-1}A_k(v_{P_k}) + PM_k^{-1}f_k, \\ -Sp_k &= J_k M_k^{-1}[A_k(v_{P_k}) + f_k] \end{aligned}$$

converge to the abstract decoupled equations?

$$\begin{aligned} \dot{v} &= \mathcal{P}\mathcal{A}(v) + \mathcal{P}f, \\ -\bar{\mathcal{J}}j\bar{\mathcal{J}}'p &= \bar{\mathcal{J}}j\mathcal{A}(v) + \bar{\mathcal{J}}jf. \end{aligned}$$

Conditions for Convergence and Sketch of Proof

(a) Space regularity

→ The differential equation is posed in H', cf. Assumption (S2), i.e., among others, at the solutions v and v_k, it holds that $\|A(v)\|_{H'}$, $\|A_k(v_k)\|_{H'} < C$, with C independent of k.

(b) Continuity and splitting properties of $\mathcal{J}$

→ The kernel of $\mathcal{J}$ splits the equation space V and the kernel of $\bar{\mathcal{J}}$ splits the equation space H

(c) Consistency and regularity

→ of the initial value and the right hand sides in the abstract equations and their discrete approximations

(d) Stable approximation schemes

→ Generally $\ker J_k \subsetneq \ker \mathcal{J}$. Thus, the dynamical equation is approximated via an external scheme. The commonly used *inf-sup* condition is sufficient for a convergence and stability of the external approximation scheme $\{V_k\}_{k\in\mathbb{N}}$ and $\{Q_k\}_{k\in\mathbb{N}}$.

(e) Continuity, Coerciveness, Monotonicity, and Boundedness of the nonlinearity

- $\rightarrow$ Requiring these properties of $\mathcal{A}$ and A_k uniformly with respect to k, we can employ the notion of *pseudomonotonicity* to show convergence of subsequences of the solutions v_{P_k} of the discrete differential equation weakly in $L^2(0, T; V)$
- $\rightarrow$ if $\mathcal{A}$ is also *weakly continuous*, then the solutions p_k of the discrete PPE converge to the solution of the continuous PPE weakly in $L^2(0, T; Q)$
- $\rightarrow$ By compactness, we can obtain strong convergence in $L^2(0, T; H)$ or $L^2(0, T; Q_H)$

Related Work and References 2

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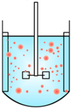

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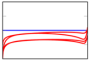

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The Abstract NSE

$$v(t) \in V,\ \dot v(t) \in V',\ p(t) \in Q_H,$$
$$\dot v - \mathcal{A}v - J'p = f,$$
$$\mathcal{J}v = 0$$

For a domain $\Omega \subset \mathbb{R}^d$, $d = \{2, 3\}$ and with the choice of

- $V := [W_0^{1,2}(\Omega)]^d$, $H := [L^2(\Omega)]^d$, and $Q_H := L^2(\Omega)/\mathbb{R}$

and

- $\mathcal{A} := \Delta\colon V \to V' := [W^{-1,2}]^d$
- $\mathcal{J} := \operatorname{div}\colon V \to Q_H' := (L^2(\Omega)/\mathbb{R})'$
- $\mathcal{J}' := \nabla\colon Q_H' \to V'$

the ADAE becomes the Stokes equation.