Practical Exercise 2 – due on June 26th

Consider the convection diffusion problem

$$\frac{\partial \rho}{\partial t} + U^T \operatorname{grad} \rho - \operatorname{div}(\mu \operatorname{grad} \rho) = 0 \quad \text{on } \Omega \tag{1a}$$

with boundary and initial conditions

$$\rho\big|_{\Gamma_1} = g(x), \quad \rho\big|_{\Gamma_4} = 2 \cdot g(y), \quad \frac{\partial \rho}{\partial n}\big|_{\Gamma_2 \cup \Gamma_3 \cup \Gamma_5} = 0 \tag{1b}$$

and

$$\rho\big|_{t=0} = \rho_0, \tag{1c}$$

with the domain Ω and its boundary $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_5$ as depicted in Figure 1 and with the velocity field

$$U = \begin{bmatrix} u_x \\ u_y \end{bmatrix} = U_0 \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} r\cos 2\phi \\ -r\sin 2\phi \end{bmatrix}, \tag{2}$$

defined via the parameter U_0 and the polar coordinates (r, ϕ).
As illustrated in Figure 1 the domain Ω is discretized in accordance to the two parameters N and xN. The MATLAB function `fvmgrid`, provided with the exercise, delivers for a choice of N and xN a vector *centxy*, containing the x and y coordinates of the cell nodes. The numbering of the cells is depicted in Figure 1. The cell boundaries are located half-way between the cell nodes. Only for the triangles the cell center is located directly at a cell boundary to ensure that the boundary is always orthogonal to the line connecting two neighboring cell nodes.

0. Download and install Paraview from `http://www.paraview.org/`. This is not mandatory, but you are provided with functions that export the matlab data to Paraview for easy visualization.

1. Visualize the flow field U for $U_0 = 1$ by computing the velocity vectors at the cell nodes and plotting the streamlines. (Paraview: Open .vtk-file, Filters/Alphabetical/Cell Data to Point Data, Streamline Tracker)

   1 Point [w]

2. Show that U is divergence free and reformulate (1a) as

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(U\rho - \mu \operatorname{grad} \rho) = 0. \tag{3}$$

   2 Point [ww]

3. Implement a spatial discretization of (1) using the grid provided. For the discretization of the convective term use the socalled θ-blending, that interpolates between the upwind and the central differences scheme, e.g. by using
$$F_h = \theta F_{h;UW} + (1-\theta)F_{h;CD},$$
where $F_{h;UW}$ and $F_{h;CD}$ are the upwind and central differences discretizations, respectively. Discuss consistency and stability of the space discretization.

   6 Points [tttwww +1 if you include the boundary conditions in the consistency analysis]

4. For the data $N = 90$, $xN = 10$, $U_0 = 5$, $g(z) = \max\{0, 1 - |1 - 8z|\}$ and $\mu(x, y) = 10^{-3}(2 - \sqrt{x^2 + y^2})$ compute the steady state solutions, i.e. $\frac{\partial \rho}{\partial t}$ is not present, of (1) for $\theta = \{0, 0.5, 1\}$. Visualize the solutions and explain the differences.

   4 Points [twww]

5. Implement implicit Euler's method and the implicit Trapezoidal rule to solve the time dependent spatially discretized problem. Document the numerically approximated time evolution of ρ for $t \in (0, 2]$, starting with $\rho_0 = 0$ and taking snapshots at intermediate points.

   3 Points [ttw]

6. Set $xN = 0$ and $U_0 = 1$. Use
$$\hat{\rho} := \max_{(x,y)\in(0,0.5)^2} \rho(x, y)$$
at time $t = 2$ to investigate the numerical convergence rate in space and time of the upwind and implicit Euler's method. Thereto compute $\hat{\rho}_{ref}$ by means of the 2nd order schemes with $N = 128$ and $Nt = 128$ timesteps. Then use upwind and implicit Euler to compute the error for $(N, Nt) \in (16, 32, 64, 128) \times (4, 16, 32, 64, 128)$. Explain your findings.

   4 Points [ttww]

MATLAB-functions provided:

- `fvmgrid` – computes the cell centers of the discretization and writes a grid.vtk file to be read in Paraview
- `addScal2gridvtk` – adds scalar cell values to the grid.vtk file
- `addVelVec2gridvtk` – adds vector valued cell data like velocities
- `mainS` – script to start with

MATLAB-functions you should use:

- `sparse, spalloc, spy`

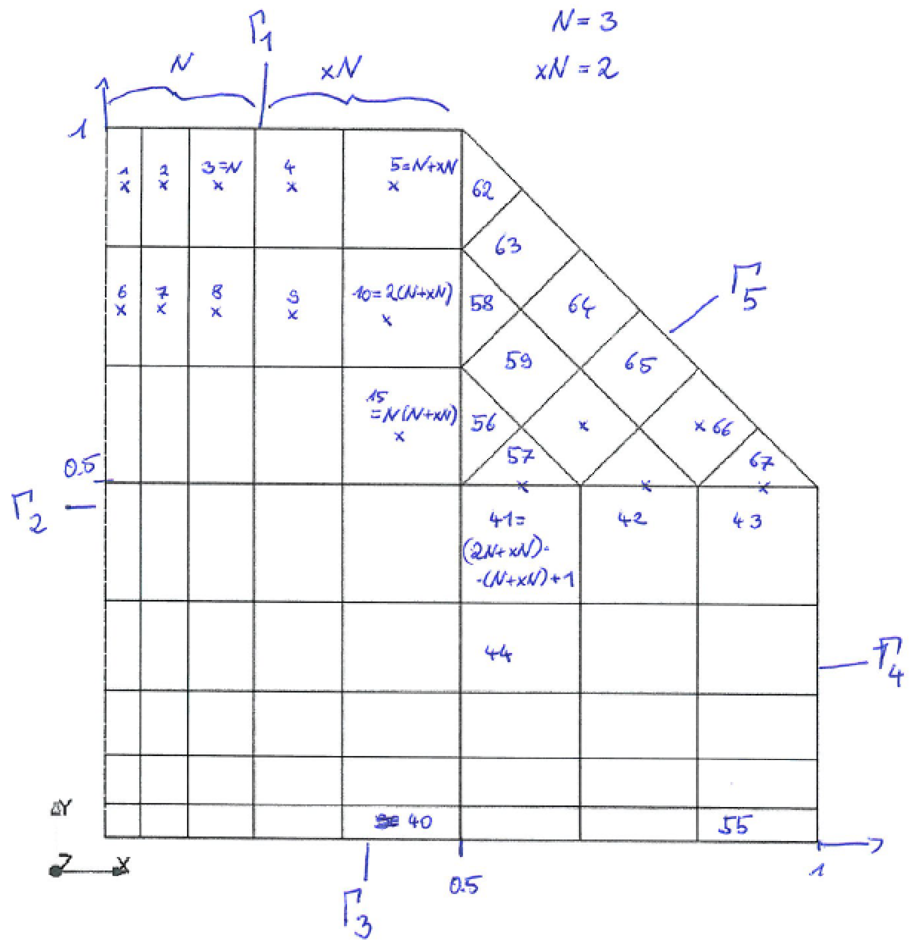

Figure 1: Illustration of the domain Ω, it's discretization by rectangular and triangular cells, the position of some exemplary cell centers and the enumeration of the cells.