May 6, 2015

Discrete Input/Output Maps and a Generalization of the Proper Orthogonal Decomposition Method

Manuel Baumann, Jan Heiland, Michael Schmidt

Conference in Honor of Volker Mehrmann's 60th Birthday, Berlin

Delft University of Technology, The Netherlands
Max Planck Institute, Magdeburg, Germany
University of Applied Sciences Offenburg, Germany

Collaborators

This is joint work of:

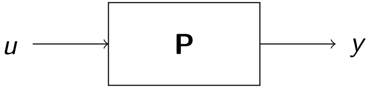
- Volker Baumann
- Volker Heiland
- Volker Schmidt

all at TU Berlin at that time.


Contents

Input-Output maps

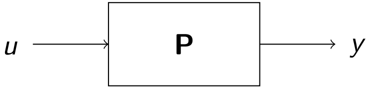
We consider the input to output (I/O) map $\mathbb{G}$ of a system **P**



that maps an input $u \in \mathcal{U}$ to the corresponding output $y \in \mathcal{Y}$.

Direct Discretization of the I/O Map

$\mathbb{G}\colon u \mapsto y$:



Idea: Discretize $\mathbb{G}$ rather than **P** (or the PDE modelling **P**)

- Focus on the relevant I/O behavior,
- which might be simple compared to the dynamics of **P**.

Discretization of I/O Maps

1. Let $\mathcal{U}$ and $\mathcal{Y}$ be Hilbert spaces and $\mathbb{G} \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$.
2. Choose subspaces $\bar{\mathcal{U}} \subset \mathcal{U}$ and $\bar{\mathcal{Y}} \subset \mathcal{Y}$ with orthogonal bases
$$\{u_1, \dots, u_{\bar{p}}\} \subset \bar{\mathcal{U}} \quad \text{and} \quad \{y_1, \dots, y_{\bar{q}}\} \subset \bar{\mathcal{Y}}.$$
3. Recall that
$$\bar{\mathcal{U}} \cong \mathbb{R}^{\bar{p}} \quad \text{and} \quad \bar{\mathcal{Y}} \cong \mathbb{R}^{\bar{q}}.$$
4. Consider the restricted and projected map
$$\mathbb{G}_S := \mathbb{P}_{\bar{\mathcal{Y}}}\mathbb{G} \in \mathscr{L}(\bar{\mathcal{U}}, \bar{\mathcal{Y}}),$$
5. which is a finite-dimensional linear map that can be expressed as a matrix
$$\mathbf{G} = \begin{bmatrix} (y_1, \mathbb{G}u_1)_{\mathcal{Y}} & \cdots & (y_1, \mathbb{G}u_{\bar{p}})_{\mathcal{Y}} \\ \vdots & \ddots & \vdots \\ (y_{\bar{q}}, \mathbb{G}u_1)_{\mathcal{Y}} & \cdots & (y_{\bar{q}}, \mathbb{G}u_{\bar{p}})_{\mathcal{Y}} \end{bmatrix} \in \mathbb{R}^{\bar{q} \times \bar{p}}.$$

Discrete I/O Maps

For a space-time tensor structure of the signal spaces,

$$\bar{\mathcal{U}} = \mathcal{R}_{\tau_1} \cdot U_{h_1} \subset L^2(0, T) \cdot U,$$
$$\bar{\mathcal{Y}} = \mathcal{S}_{\tau_2} \cdot Y_{h_2} \subset L^2(0, T) \cdot Y,$$

where U and Y are signal state spaces and with

- $\mathcal{R}_{\tau_1} = \mathrm{span}\{\phi_1, \dots, \phi_r\}$,
- $U_{h_1} = \mathrm{span}\{\mu_1, \dots, \mu_p\}$,
- $\mathcal{S}_{\tau_2} = \mathrm{span}\{\psi_1, \dots, \psi_s\}$,
- $Y_{h_2} = \mathrm{span}\{\nu_1, \dots, \nu_q\}$,

the discrete I/O map $\mathbf{G} \in \mathbb{R}^{r \times p \times s \times q}$ is a fourth order tensor.

Reduction of the I/O Map

- General purpose bases of the signal spaces may require a fine discretization,
- i.e., a high dimension of the discrete I/O map $\mathbf{G} \in \mathbb{R}^{r \times p \times s \times q}$.
- Redundancies and less important modes of $\mathbf{G}$ can be identified and truncated by means of a higher-order SVD:
  - Fix one direction, e.g. the spatial component of the output space $Y_{h_2} = \operatorname{span}\{\nu_1, \dots, \nu_q\}$.
  - Unfold the tensor into the matrix $\mathbf{G}^{(\nu)} \in \mathbb{R}^{q \times rps}$ that maps the remaining directions into Y_{h_2}.
  - Compute an SVD of $\mathbf{G}^{(\nu)}$ to find and remove redundant or almost redundant components of Y_{h_2}.
  - Do this for the other directions as well.

Relation to POD

This very idea of

1. compressing one dimension
2. using samplings of the other dimensions

is the basic principle of POD:

1. compress the spatial dimension of a state $v(t) \in \mathbb{R}^q$
2. on the base of samplings of the time dimension

via an SVD of the so called *snapshot matrix*

$$\mathbf{X} = \begin{bmatrix} v_1(t_1) & \dots & v_1(t_s) \\ \vdots & \ddots & \vdots \\ v_q(t_1) & \dots & v_q(t_s) \end{bmatrix}.$$

Generalized POD

In the particular case that

- instead of sampling in discrete instances, the *snapshots* are obtained via testing against a basis

$$\{\psi_1, \dots, \psi_s\} \quad \text{of} \quad \mathcal{S}_{\tau_2} \subset L^2(0, T),$$

- $\bar{\mathcal{Y}} = \mathcal{S}_{\tau_2} \cdot \mathbb{R}^q$, and
- the solution $\mathcal{Y} \supset v = \mathbb{G}f$ is "the output for a given input f",

the unfolded tensor I/O map

$$\mathbf{G}^{(\nu)} = \begin{bmatrix} (v_1, \psi_1)_{L^2(0,T)} & \cdots & (v_1, \psi_s)_{L^2(0,T)} \\ \vdots & \ddots & \vdots \\ (v_q, \psi_1)_{L^2(0,T)} & \cdots & (v_q, \psi_s)_{L^2(0,T)} \end{bmatrix} =: \mathbf{X}_{gen}$$

is a generalized *snapshot matrix*.

Generalized POD

Similar to the standard POD approach, one can define the POD reduced system for the generalized measurements.

Lemma

The $L^2(0, T)$-orthogonal projection $\tilde{v}(t)$ of the state vector $v(t)$ onto the space spanned by the measurements is given as

$$\tilde{v}(t) = \mathbf{X}_{gen} M_{\mathcal{S}}^{-1} \boldsymbol{\psi}(t),$$

where $\boldsymbol{\psi} := [\psi_1, ..., \psi_s]^{\mathsf{T}}$ *and where* $[M_{\mathcal{S}}]_{i,j} := (\psi_i, \psi_j)_{\mathcal{S}}$.

The generalized POD basis can be computed via a (truncated) SVD of

$$\mathbf{X}_{gen} M_{\mathcal{S}}^{-1/2}.$$

Numerical Examples

We present two test cases:

- linearized Navier-Stokes equations in 2D,
- (nonlinear) Burgers' equation in 1D,

and compare the error

$$e_{s,k} := \Big(\int_0^T \|v(t) - \tilde{v}_k(t)\|^2_{L^2(\Omega)} \mathrm{d}t\Big)^{1/2},$$

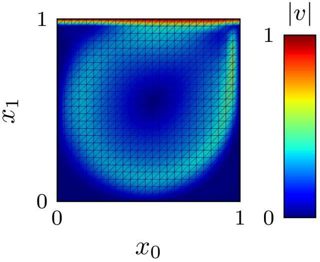
where $\tilde{v}_k$ is the solution of the system projected to the span of the k principal modes obtained through for the classical and generalized POD.

Numerical Examples

Linearized Navier-Stokes equations

Consider a model for the driven-cavity flow:

$$M\dot{v}(t) = A(\alpha, Re)v(t) + J^{\mathsf{T}}p(t) + f(t),$$
$$0 = Jv(t),$$
$$v(0) = \alpha,$$



for $Re = 2000$ and where α is the steady state Stokes solution.

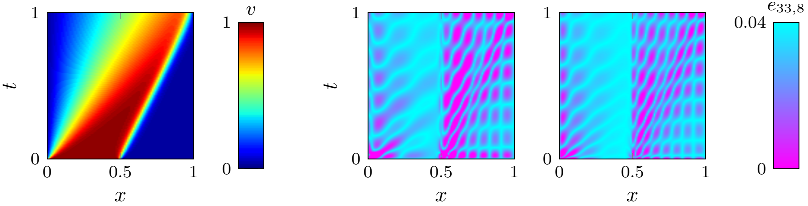
Numerical Examples

Nonlinear Burgers' equation

Consider a Burgers' equation

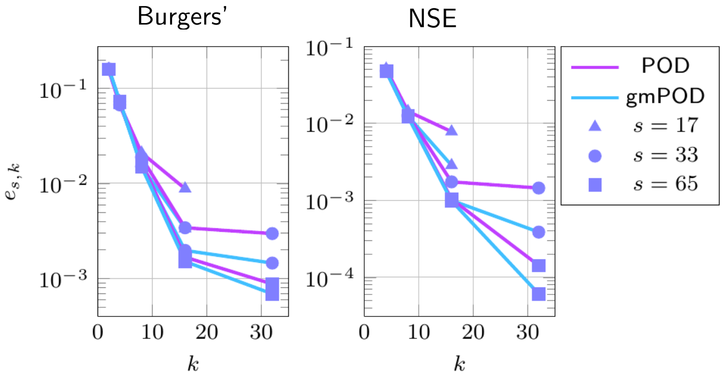
$$\partial_t z(t,x) + \partial_x\big(\frac{1}{2}z(t,x)^2 - \nu\partial_x z(t,x)\big) = 0,$$

with initial step function and homogeneous Dirichlet BC.



Error for both reduced models

L^2-error for s snapshots and POD-dimension k:



Summary

- Direct discretization of I/O maps can be further reduced through higher-order SVDs.
- From a different perspective, a reduction of a particular I/O map is a generalized version of standard POD.
- This generalization comes with two immediate advantages:
  - the measurements lie in the same space as the solution and
  - the measurements average the information.
- In the considered examples the generalized POD measurably outperformed the classical approach.
- Further work will be directed towards:
  - specific choices of the measurement functions,
  - the effect of the averaging on noisy data, and
  - error estimates for POD in the new functional setting.

Thank you!

Thank you, Volker, for everything

and thank you, the audience, for your attention!

Further reading:

M. Baumann, J. Heiland, and M. Schmidt. (2015)
Discrete Input/Output Maps and their Relation to Proper Orthogonal Decomposition.
Numerical Algebra, Matrix Theory, Differential-Algebraic Equations, and Control Theory. A Festschrift in Honor of Volker Mehrmann, Springer-Verlag

Further coding:

`www.github.com/ManuelMBaumann/genpod`